THESIS

Rigorous Results on Random Spin Systems with Competing Interactions

Hidetoshi Nishimori

Department of Physics
University of Tokyo

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Abstract

We prove various rigorous results on random spin systems. For classical models, the exact value of the internal energy is calculated in a subspace of the phase diagram, the specific heat is proved to be finite across phase boundaries and many useful relations are established on correlation functions. A local transformation of variables, the gauge transformation, is shown to play a crucial role in deriving these results. Importance of quantum effects is also pointed out. The ground-state spin quantum number is calculated for a class of quantum random spin systems.

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List of papers submitted for the requirement of the degree

1. H. Nishimori, J. Phys. C **13** (1980) 4071.
   "Exact results and critical properties of the Ising model with competing interactions"

2. H. Nishimori and M. Suzuki, Phys. Lett. **81** A (1981) 84.
   "Absence of spontaneous magnetization in spin systems with competing interactions"

3. H. Nishimori, J. Stat. Phys. (in press)
   "Spin quantum number in the ground-state of the Mattis-Heisenberg model"

4. H. Nishimori, Prog. Theor. Phys. (in press)
   "Internal energy, specific heat and correlation function of the bond-random Ising model"

List of papers added for references

1. S. Miyashita, H. Nishimori, A. Kuroda and M. Suzuki, Prog. Theor. Phys. **60** (1978) 1669.
   "Monte Carlo simulation and statistic and dynamic critical behavior of the plane rotator model"

2. H. Nishimori, Physica **97** A (1979) 589.
   "Transition point of the five-state discrete vector model"

3. H. Nishimori, J. Phys. C **12** (1979) L641.
   "Transition point of the random Ising model with bond dilution"

4. H. Nishimori, J. Phys. C **12** (1979) L905.
   "Conjecture on the exact transition point of the random Ising ferromagnet"

5. F. Tanaka and H. Nishimori, J. Phys. F **11** (1981) 1237.
   "Distribution of the metastable energy levels and the internal magnetic fields in spin glasses"

Contents

Chapter 1

Introduction

Random spin systems are of current interest both experimentally and theoretically. In random spin systems the strength of interaction between localized electron spins is not spatially uniform. In most experimental situations this randomness is quenched, that is, it is fixed at each place within the time scale of experimental relevance. Spins experience thermal fluctuations in a given configuration of randomness. If the sign of interaction between two spins is not spatially definite, the ordered state at low temperatures may become something quite different from conventional ones like ferromagnetic or antiferromagnetic states. The low temperature state of such random spin systems is usually called the spin glass state.

Experimentally a group of metallic dilute alloys like AuFe, CuMn and insulators such as $Eu_xSr_{1-x}S$ are believed to have the spin glass phase (for experimental survey of spin glass, see Mydosh 1977 [33], and Maletta and Felsch 1980 [25]). In these materials the interaction strength between localized spins is randomly distributed around zero for some reasons (*e. g.* the oscillatory nature of the RKKY interaction). The most outstanding fact indicating the existence of a phase transition in these substances is a sharp cusp in the linear magnetic susceptibility. The specific heat of the spin degree of freedom does not have any remarkable singular structure at the transition point defined by the position of cusp in the susceptibility. As is anticipated from the finiteness of the linear susceptibility, no spontaneous magnetization is observed. The cusp of linear susceptibility is very sensitive to magnetic field and is easily suppressed by a very weak external field. This sensitiveness of the susceptibility to magnetic field indicates a divergence of the non-linear susceptibility, which was directly confirmed (Chikazawa *et al*, 1980 [6]). These experimental observations characterize the spin glass phase transition. All of the above facts suggest a qualitative difference of the spin glass transition from conventional second order phase transitions (in which divergences in the specific heat and the linear susceptibility characterize the instability at the critical point).

Absence of transitional invariance evidently makes the problem non-trivial. The spatial randomness is the very cause that prevents theorists from presenting a satisfactory explanation of the phenomena. A physical picture of the spin glass state was first proposed by Edwards and Anderson (1975) [8]. They supposed that a spin in the spin glass phase has a fixed direction to point to and the direction does not change in time. The randomness is reflected in the spatial non-uniformity of the direction, yielding the vanishing total spontaneous magnetization. In other words the spin glass state is an ordered state in the

time axis but is disordered spatially. This physical picture lead Edwards and Anderson to a kind of mean field approximation and they succeeded in showing the existence of phase transition in a random spin system. Almost simultaneously Matsubara and Sakata (1976 [27]) solved a similar problem in the Bethe approximation to indicate a possibility of a new phase different from conventional ordered states. These early successes stimulated many physicists and a surge of papers followed. Among them the following contributions may be worthy to be listed here: The infinite-ranged model (Sherrington and Kirkpatrick, 1975 [37]), a phenomenological argument (Suzuki, 1977 [38]), an analysis on the Bethe lattice (Katsura and Fujiki, 1979 [21]) and numerical experiments (Binder and Schröder, 1976 [4]). All these theories more or less aimed to remedy the approximate nature of early developments. But none of them gave a conclusive explanation of the experimental situations because each of them had only a limited region of applicability.

In 1977, Toulouse [43] (inspired by Anderson) proposed a new concept of "frustration". In terms of frustration he provided a criterion to distinguish relevant randomness from irrelevant one to the spin glass state. This was a remarkable achievement because it was first recognized explicitly that the spatial non-uniformity is not a sufficient condition for the spin glass state to exist. The effect of frustration was emphasized to be the essential ingredient of the new phase. However Toulouse did not demonstrate practical usefulness of the concept of frustration in analyzing random spin systems. A few papers followed the same line of consideration (e.g. Villain 1977a [45], 1977b [46], Vanniminus and Toulouse 1977 [44]), but they are still unsatisfactory in generality or rigor which are the most important features of symmetry considerations like frustration.

The aim of the present contribution is to prove rigorous results through maximal use of symmetries in random spin systems. We provide various rigorous restrictions on the thermodynamic behavior of physical variables and on phase diagrams. Quantum effects on the symmetry in random systems are also investigated.

The construction of this thesis is as follows. In the next chapter the concept of frustration is defined and the physical significance of it is briefly discussed. A gauge transformation is a change of variables and is shown to be closely related to frustration. Formal similarity in symmetry between random spin systems and the lattice gauge theory is pointed out. Various rigorous results are derived in chapter 3 by making use of gauge transformations. Quantum effects may be extremely important in spin glass because our physical picture of the low temperature phase has its basis on analyses of the ground-state. The ground-state of a random quantum system is rigorously investigated in chapter 4. Summary and conclusion are found in the last chapter.

Chapter 2

Frustration and gauge transformation

2.1 Preliminary remarks

This chapter is devoted to definitions of basic concepts concerning symmetries of random spin systems – frustration and gauge transformation. The gauge transformation defined here plays a central role throughout this thesis. It is also demonstrated that some aspects of random spin systems are in close relation with the lattice gauge theory of strong interactions.

Before going into details, some remarks should be made to clarify what are essential in the following arguments. Examples are often given on regular lattice systems because of familiarity and easiness of understanding. However spatial regularity of the lattice is quite irrelevant to the present theory. Unless mentioned otherwise, any part of chapters 2 and 3 is valid for lattices without translational invariance. Except in chapter 4, we focus our attention on classical spin systems. Quantum effects need special considerations and we neglect them in this and the next chapters. The randomness is assumed to be quenched and therefore the configurational average should be performed to obtain observed values of thermodynamic quantities.

2.2 Frustration

We are interested in a classical spin system with the Hamiltonian

$$H = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j - h \sum_i S_i, \tag{2.1}$$

where the exchange (or maybe RKKY) interaction J_{ij} is quenched and randomly distributed following the probability function $P(J_{ij})$. The distribution of J_{ij} of a bond is assumed to be independent of those of other bonds. The spin variable S_i is classical of an arbitrary type. If S_i is a classical vector like the classical Heisenberg spin, $S_i S_j$ in (2.1) should be taken as the inner product and the external field h is supposed to be applied along one direction of spin space. The lattice structure and the range of interactions are arbitrary.

The concept of frustration is defined as follows.

Definition Consider a closed loop made up by connecting bonds on the lattice. We call $f(\text{loop})$ defined below the *frustration function* (Toulouse, 1977 [43]):

$$f(\text{loop}) = \text{sgn}\left(\prod_{\text{loop}} J\right), \tag{2.2}$$

where all interactions J_{ij} on the loop are multiplied. If $f = -1$, the bond configuration along the loop is said to be *frustrated*. If $f = +1$, it is *unfrustrated*. We exclude the case of $\prod J = 0$, for then the loop is not closed (*i. e.* at least one of J_{ij}'s on the loop is vanishing).

For instance, a plaquette (one of the smallest loops) on the square lattice is unfrustrated if all four bonds have positive interactions (Figure 2.1a). If one of the bonds is negative, the plaquette is frustrated (Figure 2.1b). If a lattice is a regular one, frustration of a plaquette is the most fundamental notion because any larger loops can be expressed as a product of appropriate plaquettes; for example, frustration of the loop surrounding plaquettes 1 and 2 in Figure 2.2 is expressed by the product of frustration functions of plaquettes 1 and 2,

$$\text{sgn}(J_{1\text{a}}J_{1\text{b}}J_{1\text{c}}J_{2\text{a}}J_{2\text{b}}J_{2\text{c}}) = \text{sgn}(J_{1\text{a}}J_{1\text{b}}J_{1\text{c}}J)\text{sgn}(JJ_{2\text{a}}J_{2\text{b}}J_{2\text{c}}).$$

To see the physical significance of frustration, it is best to investigate the ground-state of the spin 1/2 random Ising model ($S_i = \pm 1$) with $J_{ij} = \pm J$ ($J > 0$). External field is assumed to be absent. If $f = +1$ along a loop, it is easy to see that the spin configuration at $T = 0$ is uniquely determined (except for the trivial double degeneracy). For instance, if a spin S_0 on the loop is set to the up state ($S_0 = +1$), the state of another spin S_1 on the loop is determined by $S_1 = \text{sgn}(\prod J)$ where the product is taken over all bonds from the site 0 to site 1 along the loop. Since $f(\text{loop}) = +1$, this assignment works consistently. At all bonds on the loop, $-J_{ij}S_iS_j = -|J|$ with this assignment. Thus the lowest energy state is achieved at each bond along the loop. If $f = -1$, there exists a bond at which the lowest state is not achieved because the above assignment, $S_1 = \text{sgn}(\prod J)S_0$, does not apply consistently. Such a bond is "frustrated". In Figure 2.1b, for example, if all spins at the four corners are up ($S_i = +1$), then the bond with $J_{ij} = -J$ is frustrated because $-J_{ij}S_iS_j = +J$ at the bond. If the spin at the right-top corner of the same figure is down and all the others are up, then the vertical bond on the right is frustrated.

As is evident from these examples, frustration causes unsatisfied (frustrated) bonds in the ground-state. If the spin is of the Ising type and the distribution of J_{ij} is binary ($J_{ij} = \pm J$), as was in the above examples, the unsatisfaction in the ground-state usually causes degeneracy of the ground-state. Even if the ground-state is not degenerate, the density of states near the ground-state should be large; a low-lying state is obtained by making satisfied bonds in the ground-state into unsatisfied states and unsatisfied ones into the satisfied. The energy gain by this process is canceled out by the energy loss and the whole amount of the energy increase is diminished. Thus the energy difference between the ground-state and low-lying excited states is very small. It is this degeneracy of the ground-state (or the high density of states near the ground-state) that is considered to be

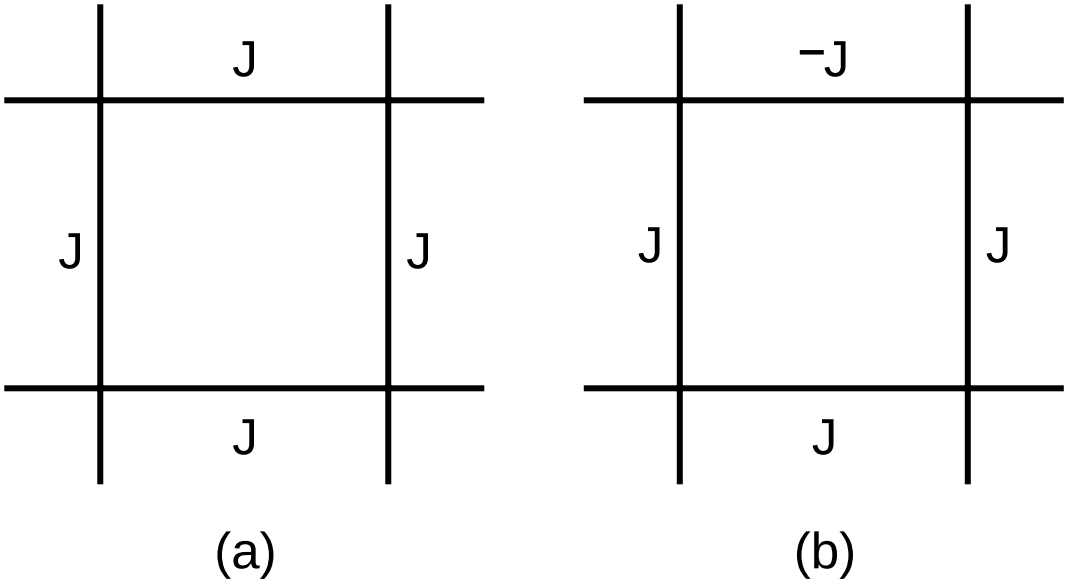

Figure 2.1:

characteristic of the spin glass state. In fact, if a system is random but free of frustrations, the ground-state is unique (except trivial degeneracy) and most of the properties of the system are reduced to those of another system with positive J_{ij}'s (see the next section).

It is interesting to further investigate how the frustration is actually connected with the spin glass phase transition (Toulous 1977 [43], Fradkin *et al* 1978 [12], Vannimenus and Toulous 1977 [44]). However our aim is not to provide a physical argument to establish a definite picture of the spin glass phase. Rather we intend to prove rigorous results with the help of a "gauge transformation" which is closely related to frustration. Consequently we think it better not to devote more pages to explanation of frustration, except for one point – irrelevance of frustration-free randomness to thermodynamic quantities. The proof of irrelevance of this type of randomness serves as a good introduction to the idea of gauge transformation.

2.3 Frustration-free systems

If any loop of a system is free of frustration, we call it a frustration-free system. From discussions in the previous section it is clear that a frustration-free system has the ground-state with all bonds in the lowest energy state. This property reminds us of ferromagnetic systems. We should in this section that a frustration-free systems is, even if the signs of J_{ij}'s are not positive definite, equivalent to a ferromagnetic system.

Proposition A frustration-free system can be written as a system with quenched randomness at sites (not at bonds).

Proof: Let us introduce a sign function τ_{ij}: $J_{ij} = \tau_{ij}|J|$. By τ_{ij} the frustration function

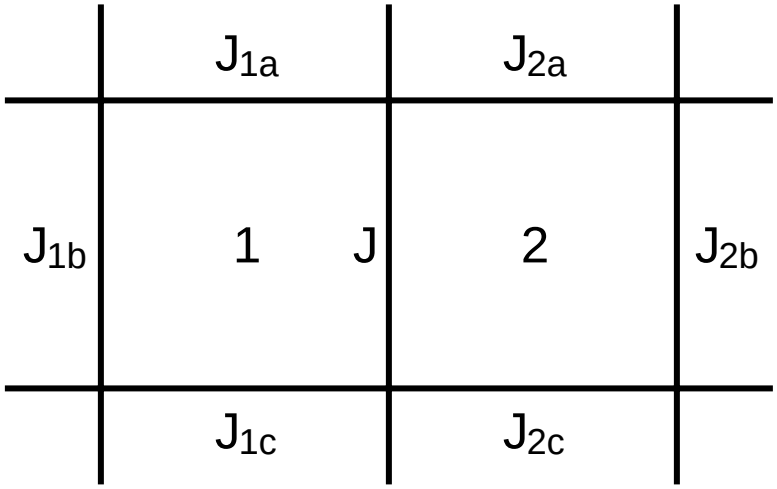

Figure 2.2:

is expressed as

$$f(\text{loop}) = \prod_{\text{loop}} \tau_{ij}. \tag{2.3}$$

If $f = +1$ for all loops, it is evident that we can assign to each site a random variable τ_i by

$$\tau_i = \left(\prod_{\Gamma} \tau_{ij} \right) \tau_0, \tag{2.4}$$

where 0 is an arbitrarily chosen site on the lattice, τ_0 is fixed to either $+1$ or -1 and Γ is an arbitrarily specified path from 0 to i (Figure 2.3). Since $f = +1$ for all loops, τ_i $(= \pm 1)$ is well-defined (τ_i does not depend on the path Γ). The original bond variable τ_{ij} is now written as $\tau_{ij} = \tau_i \tau_j$ as is clear from Eq. (2.4) and Figure 2.3. Therefore we may express the Hamilton as

$$H = -\sum_{\langle ij \rangle} |J_{ij}| \tau_i \tau_j S_i S_j - h \sum_i S_i. \tag{2.5}$$

Proposition If the randomness is allocated to sites as in (2.5) and not to bonds, the systems is free of any frustrations.

Proof: In constructing the frustration function of a loop, any random variable τ_i appears even times and hence the proposition follows.

The model systems (2.5) is called the *Mattis model* (Mattis,1976 [28]). In the absence of external field, thermodynamic properties of the Mattis model are equivalent to those of systems with all $\tau_i = +1$.

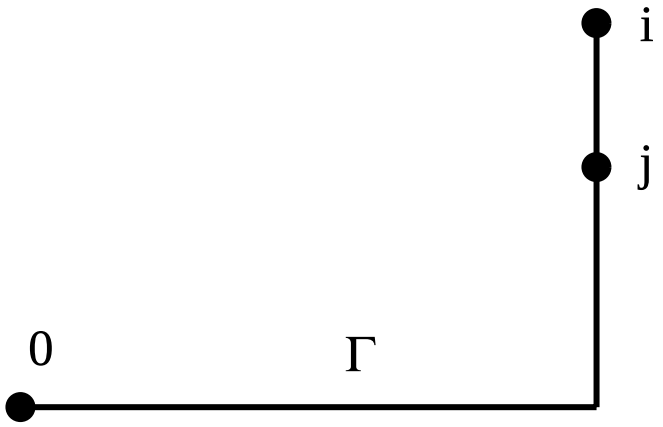

Figure 2.3:

Proposition Thermodynamic properties of the Mattis model without external field are not affected by the randomness in the configuration $\{\tau_i\}$.

Proof: The free energy of the system is calculated as

$$F = -kT \ln \mathrm{Tr} \exp \left(\beta \sum_{\langle i,j \rangle} |J_{ij}| \tau_i \tau_j S_i S_j \right).$$

We change here the signs of spin variables $\tau_i S_i \to S_i$ at all sites. The trace "Tr" of a classical spin system is invariant by this change of variables and therefore

$$F = -kT \ln \mathrm{Tr} \exp \left(\beta \sum_{\langle i,j \rangle} |J_{ij}| S_i S_j \right).$$

The three propositions given above indicate that a frustration-free system cannot be a candidate of a model system of spin glass. A frustration-free system in the absence of magnetic field is essentially equivalent to a ferromagnetic system which has a clear singularity in the specific heat at a critical point. It should be noted however that the effect of external field is a non-trivial one even in the Mattis model. If we change the variables, $\tau_i S_i \to S_i$, as we did in the above proof, the term of the external field in the Hamilton (2.5) is changed into $h \sum \tau_i S_i$. Thus we cannot eliminate the randomness in the presence of h.

2.4 Gauge transformation and gauge invariance

In the previous section a transformation $\tau_i S_i \to S_i$ was used to change the Mattis model (without external field) into a ferromagnetic system. A generalization of the transformation to arbitrary random systems is found to be extremely useful in deriving rigorous

results as will be shown in chapter 3.

Definition *A gauge transformation* of the system (2.1) at site i is defined as a change of variables,

$$\begin{aligned} S_i &\to -S_i, \qquad J_{ij} \to -J_{ij} \\ (j &\text{ adjacent to } i, \textit{ i. e. } J_{ij} \neq 0.) \end{aligned} \tag{2.6}$$

Locality of the transformation (2.6) is the most important feature of it; the transformation may be performed or not independently at each site. Another expression of the transformation (2.6) is possible:

$$S_i \to \sigma_i S_i, \qquad J_{ij} \to \sigma_i \sigma_j J_{ij}, \tag{2.7}$$

where $\sigma_i = -1$ if the transformation (2.6) is performed at site i and $\sigma_i = +1$ otherwise. For instance, if the transformation is performed only at site i like (2.6), we should choose $\sigma_i = -1$ and $\sigma_j = +1$ for all j $(\neq i)$. Then (2.7) reduces to (2.6). The definition (2.7) is valid for all kinds of classical spins $\{S_i\}$, but not for quantum ones. Quantum mechanical spins are forbidden to change signs independently at each site because commutation relations are not thereby preserved. This impossibility of gauge transformation in quantum spin systems is discussed in detail in chapter 4.

Definition *A gauge invariant quantity* is defined as a physical quantity which is left unchanged any gauge transformation expressed by (2.7).

The Hamiltonian (2.1) with $h = 0$ is evidently gauge invariant. The thermal expectation value of the Hamiltonian, the internal energy, is also gauge invariant. In general thermal expectation value of a gauge invariant quantity is again gauge invariant if the external field is absent.

On the other hand correlation functions are not gauge invariant. Let us consider the thermal expectation value of an arbitrary product of spins $\langle \prod S \rangle$, where the angular brackets denote the thermal average by the Hamiltonian (2.1). In the absence of external field the Hamiltonian is invariant by the transformation (2.7) and therefore the thermal average remains unchanged. The product of spins is multiplied by $\prod \sigma$ where the product is taken over the same set of sites as in the product $\prod S$. Since σ's are arbitrarily fixed, they are independent of the thermal average. Thus $\langle \prod S \rangle$ is transformed into $\prod \sigma \langle \prod S \rangle$ which is different from $\langle \prod S \rangle$ if $\prod \sigma = -1$. Since a gauge invariant quantity should not change its sign under any gauge transformation, a correlation function cannot be a gauge invariant quantity.

It is possible to modify the definition of correlation function to make it gauge invariant. A modification was introduced by Fradkin *et al* (1978 [12]). We illustrate their idea by a two-point correlation function in the absence of external field. Fradkin *et al* defined a gauge invariant correlation function by

$$C_\Gamma(ij) = \left\langle S_i \left(\prod_\Gamma \tau \right) S_j \right\rangle, \tag{2.8}$$

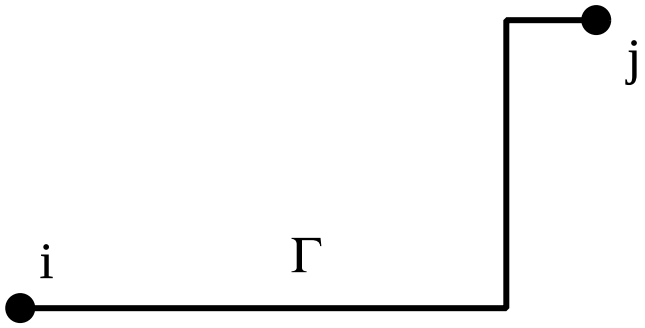

Figure 2.4:

where the product of $\tau_{kl} = (J_{kl}/|J_{kl}|)$ is taken over all τ's on an arbitrarily specified path Γ connecting i and j (Figure 2.4). To prove that C_Γ is gauge invariant, we notice that any σ introduced by the transformation (2.7) appears even times in Γ; if a site k is on the path (including the edges i and j), then σ_k appears twice. Otherwise σ_k does not appear in C_Γ after the gauge transformation. Therefore, since a $\sigma_k^2 = 1$, we conclude that a gauge invariant correlation function is in fact gauge invariant. Physical significance and thermodynamic behavior of the gauge invariant correlation functions are discussed in chapter 3.

2.5 Comment on the lattice gauge theory

The idea of gauge transformation, particularly in the form (2.7), originates in the lattice gauge theory of strong interactions. To see the similarities between random spin systems and the lattice gauge theory, we briefly review the mathematical structure of the lattice gauge theory (Kogut, 1979 [22]). Since it is not our purpose to explain detailed physical backgrounds of the lattice gauge theory, we start off with writing down the vacuum to vacuum transition probability of a system of pure gauge fields with the simplest symmetry Z_2, the symmetry of the Ising model [1]:

$$Z = \sum_{\{S=\pm 1\}} \exp\left(\frac{1}{g^2}\sum S_1 S_2 S_3 S_4\right), \tag{2.9}$$

where g is the coupling constant (corresponding formally to the temperature in statistical mechanics) and the field variables $\{S_i\}$ are allocated to *bonds* of a d-dimensional hypercubic lattice. The product of variables is over four S_i's surrounding a plaquette (Figure 2.5). In the exponent of (2.9) the product of four S_i's are summed over all plaquettes on the lattice. The vacuum to vacuum transition probability (2.9) occupies the same position as the partition function does in statistical mechanics. For lattices with $d > 2$, two types of phases are supposed to exist. The two phases are distinguished from each other by the

[1] Only more complicated symmetries such as $SU(N)$ have direct physical significance. But the mathematical structure we are interested in is described by Z_2 well enough.

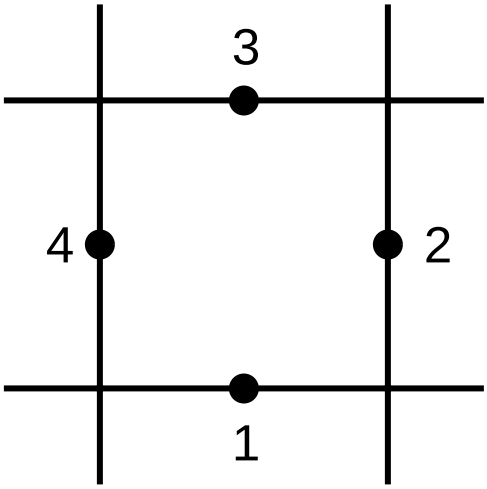

Figure 2.5:

difference in the asymptotic behavior of the expectation value of the "Wilson loop",

$$\left\langle \prod_{\text{loop}} S \right\rangle \sim \begin{cases} \exp(-aL) & (2.10) \\ \exp(-bA), & (2.11) \end{cases}$$

where S_i's are multiplied along an asymptotically large loop (Wilson, 1974 [47]) on the lattice. The average $\langle \cdots \rangle$ is performed in quite the same way as in statistical mechanics by the partition function (2.9). If the expectation value decays by the perimeter law (2.10) (L is the length of the loop), then the system is in the small-g phase. In this phase the source matter of the gauge field is considered to exist as a single particle. On the other hand, if the asymptotic behavior obeys the area law (2.11) (A denotes the area surrounded by the loop), then the system is in the large-g phase which is the confinement region. Roughly speaking, if the source particle of the gauge field is confined, the system cannot sustain a separation of the particles by a large distance and the expectation value decays rapidly as in (2.11). A sharp transition between these qualitatively different asymptotic behaviors is signalized also by a singularity in the partition function (2.9) as is usual in phase transitions of ordinary spin systems. Thus the mathematical techniques developed for statistical mechanics, particularly for critical phenomena, are useful to study strong interactions.

We now demonstrate the gauge invariance of the partition function (2.9). A gauge transformation in this system is performed at each site of the lattice (note that the field variables are on bonds) . A gauge transformation is defined as a change of signs of all S_i's on bonds emenating from a site (Figure 2.6). As a result negative signs appear even times in the combination $S_1S_2S_3S_4$ which is thus left invariant. It is seen that the invariance here has a very similar structure to the invariance of the Hamiltonian (2.1) with $h = 0$ under the gauge transformation (2.6). This similarity can be made use of in our investigation of the two theories (Fradkin et al , 1978 [12]). An outstanding example is found in section 7 of chapter 3 where the absence of ferromagnetic long range order is proved by using the local gauge transformation – the same idea as the proof of the Elitzur's theorem of the

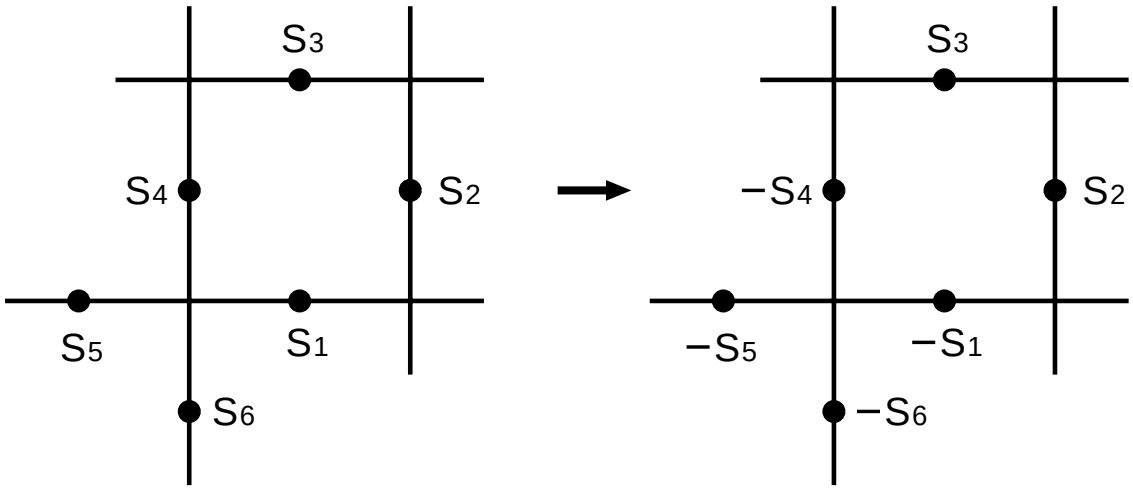

Figure 2.6:

lattice gauge theory (Elitzur 1975 [9], Kogut 1979 [22]).

Chapter 3

Physical results obtained by gauge transformations

In this chapter we provide applications of gauge transformations to proofs of various rigorous results on random spin systems. We show that it is possible to derive rigorous relations on the internal energy, specific heat and correlation functions. Section 3.1 is devoted to a general preparation. In sections from 3.2 to 3.5 we treat the spin 1/2 random Ising model without external field. In sections 3.6 and 3.7, response to external field is investigated for arbitrary classical spin systems. We prove the existence of ferromagnetic phase transition in section 3.8.

3.1 Configurational average expressed by the partition function

The observed value of a physical quantity is obtained by averaging the thermal expectation value over a distribution of $\{J_{ij}\}$. This is called the configurational average which we symbolize by $[\cdots]$ to distinguish from the thermal average $\langle\cdots\rangle$. For instance, the configurational average of the free energy is written as

$$[F] = -kT \int_{-\infty}^{\infty} \prod_{(i,j)} \mathrm{d}J_{ij} P(J_{ij}) \ln \mathrm{Tr}_S \exp\left(\beta \sum_{(i,j)} J_{ij} S_i S_j\right). \tag{3.1}$$

Let us rewrite the configurational average, the integration over $\{J_{ij}\}$ in (3.1), into a more convenient form. We restrict the range of integrations to positive values and introduce another set of variables $\{\tau_{ij}\}$ to cover the negative range:

$$\begin{aligned}[F] &= -kT \int_{0}^{\infty} \prod_{(i,j)} \left\{ \mathrm{d}J_{ij} \frac{P(J_{ij}) + P(-J_{ij})}{2\cosh A_{ij}} \right\} \\ &\quad \times \sum_{\{\tau_{ij}=\pm 1\}} \exp\left(\sum_{(i,j)} \tau_{ij} A_{ij}\right) \ln \mathrm{Tr}_S \exp\left(\beta \sum_{(i,j)} J_{ij} \tau_{ij} S_i S_j\right). \end{aligned} \tag{3.2}$$

Here A_{ij} is defined by

$$\exp(-2A_{ij}) = \frac{P(-J_{ij})}{P(J_{ij})} \quad (J_{ij} > 0). \tag{3.3}$$

By (3.3) the correct relative weight of negative ($\tau = -1$) and positive ($\tau = +1$) bonds is assigned in (3.2). The factor $2\cosh A_{ij}$ in (3.2) is the normalization of the weighted sum over $\{\tau_{ij}\}$[1]. Now perform a gauge transformation in (3.2),

$$S_i \to S_i\sigma_i, \qquad \tau_{ij} \to \sigma_i\sigma_j\tau_{ij}. \tag{3.4}$$

Since the summations over $\{\tau_{ij}\}$ and $\{S_i\}$ (the latter symbolized as Tr_s in (3.2)) are apparently invariant under the gauge transformation, we find

$$\begin{aligned}\big[F\big] &= -kT\int_0^\infty \prod_{(i,j)} \mathrm{d}J_{ij} Q(J_{ij}) \sum_{\{\tau_{ij}=\pm 1\}} \exp\left(\sum_{(i,j)} \tau_{ij}A_{ij}\sigma_i\sigma_j\right) \\ &\quad \times \ln \mathrm{Tr}_S \exp\left(\beta \sum_{(i,j)} J_{ij}\tau_{ij}S_iS_j\right),\end{aligned} \tag{3.5}$$

where we have used an abbreviation $Q(J_{ij})$ for $(P(J_{ij}) + P(-J_{ij}))/2\cosh A_{ij}$. The gauge transformation introduced above is just a redefinition of variables and has no effects on the value of the free energy. Consequently the free energy (3.5) is independent of the choice of the set $\{\sigma_i\}$. Because of this independence the following manipulations leave the free energy invariant: Sum (3.5) over all possible configurations of $\{\sigma_i\}$ and then divide by 2^N (the total number of configurations of $\{\sigma_i\}$). In this way we have

$$\begin{aligned}\big[F\big] &= -kT\int_0^\infty \prod_{(i,j)} \mathrm{d}J_{ij}\, Q(J_{ij})\, 2^{-N} \sum_{\{\tau_{ij}=\pm 1\}} \sum_{\{\sigma_{ij}=\pm 1\}} \exp\left(\sum_{(i,j)} \tau_{ij}A_{ij}\sigma_i\sigma_j\right) \\ &\quad \times \ln \mathrm{Tr}_S \exp\left(\beta \sum_{(i,j)} J_{ij}\tau_{ij}S_iS_j\right).\end{aligned} \tag{3.6}$$

Let us compare this equation with (3.2). In (3.2) the relative weight factor of positive and negative bonds was given by $\exp(\sum \tau_{ij}A_{ij})$. In (3.6) the same quantity is expressed

[1]To verify that $2\cosh A_{ij}$ is the normalization factor, we calculate the configurational average of 1:

$$\begin{aligned}[1] &= \int_0^\infty \prod_{(i,j)} \left\{\mathrm{d}J_{ij} \frac{P(J_{ij}) + P(-J_{ij})}{2\cosh A_{ij}}\right\} \sum_{\tau_{ij}=\pm 1} \exp\left(\sum_{(i,j)} \tau_{ij}A_{ij}\right) \times 1 \\ &= \int_0^\infty \prod_{(i,j)} \left\{\mathrm{d}J_{ij} \frac{P(J_{ij}) + P(-J_{ij})}{2\cosh A_{ij}}\right\} \prod_{(i,j)} 2\cosh A_{ij} \\ &= \prod_{(i,j)} \int_{-\infty}^\infty \mathrm{d}J_{ij} P(J_{ij}) = 1.\end{aligned}$$

by the partition function of a random Ising model with the Hamiltonian

$$H = -\sum_{(i,j)} \tau_{ij} A_{ij} \sigma_i \sigma_j$$

on the same lattice as our original systems (2.1). (A similar result has been obtained by Fradkin *et al* (1978 [12]).) This observation yields numerous rigorous results.

3.2 Internal energy

By making use of (3.6), we can calculate the exact value of the internal energy in a subspace of the phase diagram. Some interesting examples will follow a general discussion. Only the spin l/2 Ising system ($S_i = \pm 1$) is treated in this section. Correspondingly the trace over $\{S_i\}$ is written as $\sum_S$.

3.2.1 General expression

The internal energy is calculated by differentiating the free energy,

$$\begin{aligned}
\big[\,\langle E\rangle\,\big] &= \frac{\partial\big[\beta F\big]}{\partial\beta} \\
&= -\int_0^\infty \prod \mathrm{d}J_{ij}\, Q(J_{ij})\, 2^{-N} \sum_\tau \sum_\sigma \exp\Big(\sum \tau_{ij} A_{ij} \sigma_i \sigma_j\Big) \\
&\quad \times \frac{\dfrac{\partial}{\partial\beta} \displaystyle\sum_S \exp\Big(\beta \sum J_{ij} \tau_{ij} S_i S_j\Big)}{\displaystyle\sum_S \exp\Big(\beta \sum J_{ij} \tau_{ij} S_i S_j\Big)}.
\end{aligned} \tag{3.7}$$

Since both σ and S are Ising variables, it is seen in (3.7) that, if $A_{ij} = \beta J_{ij}$, a cancellation of the two partition functions, one with variables $\{\sigma_i\}$ and the other with $\{S_i\}$, happens:

$$\big[\,\langle E\rangle\,\big] = -\int_0^\infty \prod \mathrm{d}J_{ij}\, Q(J_{ij})\, 2^{-N} \sum_\tau \frac{\partial}{\partial\beta} \sum_S \exp\Big(\beta \sum J_{ij} \tau_{ij} S_i S_j\Big)\Big|_{A_{ij}=\beta J_{ij}}.$$

This equation is easily evaluated. We sum over $\{\tau\}$ and then over $\{S\}$,

$$\begin{aligned}
\big[\,\langle E\rangle\,\big] &= -\int_0^\infty \mathrm{d}J_{ij}\, Q(J_{ij})\, 2^{-N} \frac{\partial}{\partial\beta} \sum_S \prod_{(i,j)} 2\cosh\beta J_{ij}\Big|_{A_{ij}=\beta J_{ij}} \\
&= -\frac{\partial}{\partial\beta} \int_0^\infty \prod \mathrm{d}J_{ij}\, Q(J_{ij}) \prod 2\cosh\beta J_{ij}\Big|_{A_{ij}=\beta J_{ij}}.
\end{aligned}$$

Here the integrations decouple as

$$\big[\,\langle E\rangle\,\big] = -\frac{\partial}{\partial\beta} \prod_{(i,j)} \int_0^\infty \mathrm{d}J_{ij} \frac{(P(J_{ij}) + P(-J_{ij}))2\cosh\beta J_{ij}}{2\cosh A_{ij}}\Big|_{A_{ij}=\beta J_{ij}},$$

where we have rewritten $Q(J_{ij})$ into the original definition. It is now possible to carry out the differentiation by β and then set A_{ij} equal to βJ_{ij}:

$$\big[\,\langle E\rangle\,\big] = -\sum_{(i,j)}\int_0^\infty \mathrm{d}J_{ij}(P(J_{ij})+P(-J_{ij}))J_{ij}\tanh\beta J_{ij}. \tag{3.8}$$

The quantity $\tanh\beta J_{ij}$ in the integrand of (3.8) can be put into a simpler form,

$$\begin{aligned}\tanh\beta J_{ij} &= \frac{1-\exp(-2\beta J_{ij})}{1+\exp(-2\beta J_{ij})} = \frac{1-P(-J_{ij})/P(J_{ij})}{1+P(-J_{ij})/P(J_{ij})} \\ &= \frac{P(J_{ij})-P(-J_{ij})}{P(J_{ij})+P(-J_{ij})},\end{aligned} \tag{3.9}$$

where we have used the definition (3.3) of A_{ij} and the fact $A_{ij}=\beta J_{ij}$. By replacing $\tanh\beta J_{ij}$ in (3.8) by (3.9), we find

$$\begin{aligned}\big[\,\langle E\rangle\,\big] &= -\sum_{(i,j)}\int_0^\infty \mathrm{d}J_{ij}\,\{P(J_{ij})-P(-J_{ij})\}\,J_{ij} \\ &= -\sum_{(i,j)}\int_0^\infty \mathrm{d}J_{ij}\,J_{ij}P(J_{ij}) \\ &= -\sum_{(i,j)}\big[J_{ij}\big] = -\frac{1}{2}zN\big[J_{ij}\big],\end{aligned} \tag{3.10}$$

where z is the coordination number (and consequently $zN/2$ is the total number of interacting pairs). If the lattice is not a regular one, z is the average number of nearest neighbors.

Before going into detailed discussions on the relation between the phase diagram and the internal energy (3.10), we give a comment on the condition $A_{ij}=\beta J_{ij}$ under which (3.10) was derived. The relation $A_{ij}=\beta J_{ij}$, or equivalently $\exp(2\beta J_{ij})=P(-J_{ij})/P(J_{ij})$, suggests us to try factorizing $P(J_{ij})$ as

$$P(J_{ij}) = \exp(aJ_{ij})f(J_{ij}). \tag{3.11}$$

In this expression the condition $A_{ij}=\beta J_{ij}$ is seen to be satisfied if $a=\beta$ and $f(J_{ij})$ is an even function of J_{ij}. Of course the probability distribution (3.11) is originally independent of temperature, but we have to put $a=\beta$ to obtain (3.9). The constraint $a=\beta$ is equivalent to confining ourselves to a subspace in the phase diagram because the constraint reduces a degree of freedom. This reduction of a degree of freedom is well illustrated in examples presented later.

To summarize; if the distribution function of J_{ij} has the form of (3.11) with $f(J_{ij})$ an even function, the internal energy can be calculated exactly in a subspace (defined by $a=\beta$) in the phase diagram and the result is (3.10).

3.2.2 Example 1: Gaussian distribution

The Gaussian distribution

$$P(J_{ij}) = \frac{1}{\sqrt{2\pi}J}\exp\left(-\frac{(J_{ij}-J_0)^2}{2J^2}\right) \tag{3.12}$$

was first adopted by Edwards and Anderson (1975 [8]) in their theory of spin glass. Sherrington and Kirkpatrick (1975 [37]) tried to solve an "exactly soluble model" which has infinitely long-ranged interactions and the Gaussian distribution function (3.12). Ever since the appearance of these papers, the distribution (3.12) has been repeatedly used in the literature because, first, it is often the most plausible assumption for unknown stochastic quantities and, second, calculations are sometimes greatly simplified by the Gaussian assumption.

The Gaussian function (3.12) can be put into the form of (3.11):

$$P(J_{ij}) = \exp\left(\frac{J_0 J_{ij}}{J^2}\right) \frac{\exp\left(-\dfrac{J_{ij^2}}{2J^2} - \dfrac{J_0^2}{2J^2}\right)}{\sqrt{2\pi}J}.$$

Thus the condition $a = \beta$ for (3.10) to be valid is $J_0/J^2 = \beta$ or $\beta J^2 = J_0$. Since the phase diagram of the present model is usually drawn in terms of the axes $(J_0/J,\ kT/J)$ as in Figure 3.1, the constraint $\beta J^2 = J_0$ (or equivalently $J_0/J = J/kT$) confines us to a hyperbolic curve in the phase diagram. On the hyperbolic curve the internal energy is

$$\big[\,\langle E\rangle\,\big] = -\frac{1}{2}zN\big[J_{ij}\big] = -\frac{1}{2}zNJ_0 \tag{3.13}$$

which shows no singular behavior. It should be emphasized that this absence of singularities in (3.13) never implies the absence of phase boundaries across the curve $\beta J^2 = J_0$ on which we have obtained (3.13). There may occur a cancellation of singularities along the curve $\beta J^2 = J_0$. A possible mechanism of cancellation is illustrated by the long-range limit of interactions.

The long-range limit of interactions is of particular interest because Sherrington and Kirkpatrick (1975 [37]) solved it exactly (although somewhat problematic limiting procedures were made use of). If the interactions are long-ranged, we have to rescale the parameters in (3.12) to keep physical quantities extensive. In order for the Hamiltonian to be extensive, the typical value J_0 of the interaction should be proportional to N^{-1}: $J_0 = \tilde{J}_0/N$ ($\tilde{J}_0$ independent of N). The fluctuation J is proportional to $\sqrt{N}^{-1}$, $J = \tilde{J}/\sqrt{N}$, so that the mean fluctuation of the Hamiltonian $\langle (H - \langle H\rangle)^2\rangle$ is proportional to N. The constraint $\beta J^2 = J_0$ is now written as $\beta\tilde{J}^2 = \tilde{J}_0$. The number of interacting pairs is equal to $N(N-1)/2 \sim N^2/2$ (if N is large enough). Thus we have the internal energy, from (3.13), as

$$\big[\,\langle E\rangle\,\big] = -\frac{1}{2}\tilde{J}_0 N. \tag{3.14}$$

The above expression coincides with that of Sherrington and Kirkpatrick (1975 [37]) restricted to the line $\beta\tilde{J}^2 = \tilde{J}_0$ as is shown in the following.

The general expression of the internal energy given by them reads (their Eqn. (16))

$$\big[\,\langle E\rangle\,\big] = -N\left\{\frac{1}{2}\tilde{J}_0 m^2 + \frac{1}{2}\beta\tilde{J}^2(1-q^2)\right\}, \tag{3.15}$$

where m $(= [\langle S_i\rangle])$ and q $(= [\langle S_i\rangle^2])$ are order parameters. As is proved in Appendix, the two order parameters m and q are equal to each other on the line $\beta\tilde{J}^2 = \tilde{J}_0$. Hence, if

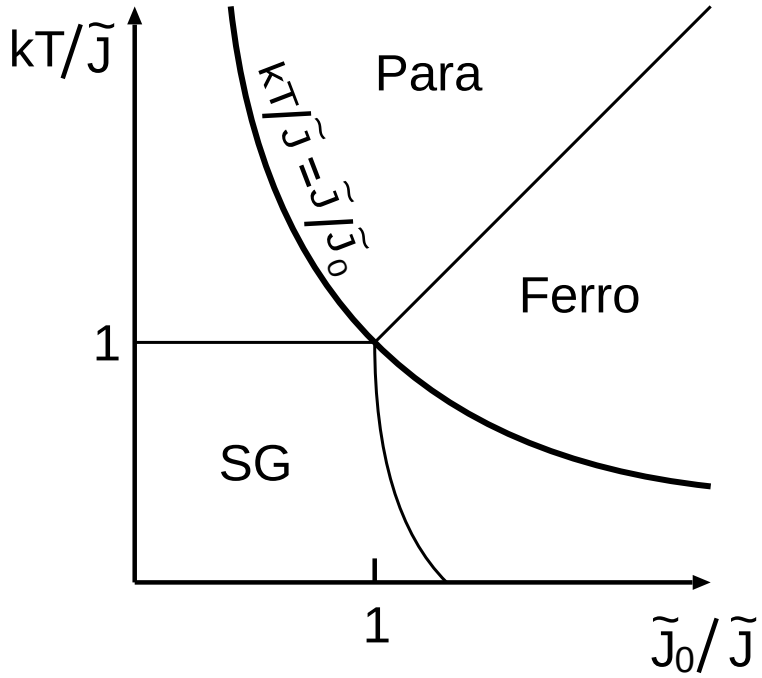

Figure 3.1: The random system with the Gaussian distribution function has a simple behavior on the hyperbolic curve in the phase digram.

$\beta\tilde{J}^2 = \tilde{J}_0$, (3.15) reduces to

$$\left[\, \langle E \rangle \,\right] = -\frac{1}{2} N \beta \tilde{J}^2 = -\frac{1}{2} N \tilde{J}_0{}^2,$$

which is exactly equal to (3.14).

The internal energy (3.14) is free of singularities, although the tricritical point (where the ferromagnetic, paramagnetic and spin glass phases merge) is on the curve $\beta\tilde{J}^2 = \tilde{J}_0$ on which we have (3.14). The mechanism of cancellation of singularities at the tricritical point is easy to see in (3.15): In (3.15) the singularity of the internal energy is present in the terms proportional to m^2 and q^2 (since these order parameters are singular at phase boundaries). However, on the line $\beta\tilde{J}^2 = \tilde{J}_0$, q is equal to m and therefore a perfect cancellation of the singular terms happens in (3.15).

3.2.3 Example 2: Binary distribution

Another interesting distribution function is the binary one:

$$P(J_{ij}) = p\delta(J_{ij} - J) + (1-p)\delta(J_{ij} + J). \tag{3.16}$$

Matsubara and Sakata (1976 [27]) adopted this distribution function to explain some of the properties of magnetic binary mixtures (Adachi *et al* 1969 [2], 1970 [1]). Later Toulouse (1977 [43]) pointed out the importance of the binary distribution to extract the frustration effect.

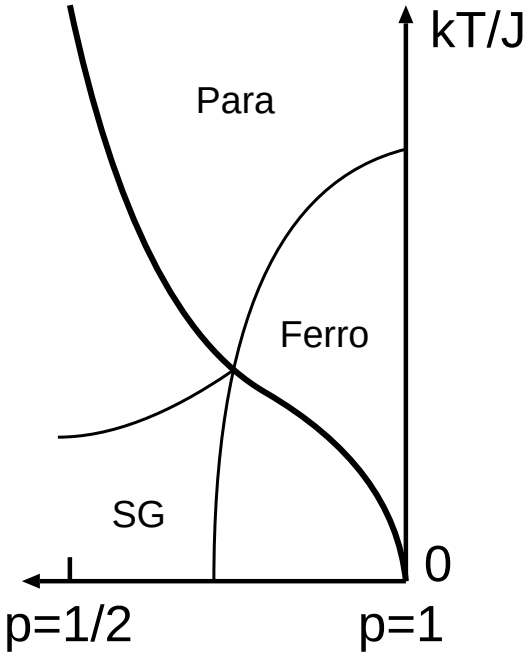

Figure 3.2: In the binary distribution case, the curve corresponding to the hyperbolic one in Figure 3.1 extends from ($T = 0$, $p = 1$) to ($T = \infty$, $p = 1/2$).

Let us change (3.16) into a form of (3.11),

$$P(J_{ij}) = \frac{\mathrm{e}^{K_p J_{ij}/J}}{2\cosh K_p} \left\{\delta(J_{ij} - J) + \delta(J_{ij} + J)\right\}, \tag{3.17}$$

where K_p is defined by

$$\exp(-2K_p) = \frac{1-p}{p}. \tag{3.18}$$

To verify that (3.17) is equivalent to (3.16) ,we first notice that (3.17) is normalized

$$\int_{-\infty}^{\infty} \mathrm{d}J_{ij} P(J_{ij}) = 1.$$

Next we see that the coefficients of the two delta functions $\delta(J_{ij} - J)$ and $\delta(J_{ij} + J)$ are $\exp(K_p)/2\cosh K_p$ and $\exp(-K_p)/2\cosh K_p$ respectively. These coefficients are easily understood to give the correct weight of positive and negative bonds by (3.18).

The condition for the formula (3.10) to be valid is now expressed as $K_p/J = \beta$, or equivalently $\exp(-2\beta J) = (1-p)/p$. This constraint restricts us to a curve in the two-dimensional phase diagram: The phase diagram of the present system is usually drawn as Figure 3.2. The condition $\exp(-2\beta J) = (1-p)/p$ provides a restriction to a curve which starts from the ferromagnetic ground-state ($T = 0$, $p = 1$) and ends at an infinitely high temperature point ($T = \infty$, $p = 1/2$). On this curve the internal energy is, according to

(3.10),

$$\begin{aligned} \left[\, \langle E \rangle \,\right] &= -\frac{1}{2} zNJ(2p-1) \\ &= -\frac{1}{2} zNJ \tanh \beta J. \end{aligned} \tag{3.19}$$

This internal energy is free of singularities although the curve $\exp(-2\beta J) = (1-p)/p$ (on which we have proved the above relation) may intersect phase boundaries (see Figure 3.2). This is not a contradiction as was shown in the previous subsection; a cancellation of singularities would have yielded the apparently regular internal energy (3.19).

3.2.4 Example 3: Diluted binary distribution (Horiguchi, 1981)

The next distribution function is

$$P(J_{ij}) = p\delta(J_{ij} - J) + r\delta(J_{ij}) + q\delta(J_{ij} + J) \quad (J > 0) \tag{3.20}$$

with $p + r + q = 1$. This distribution has attracted much attention (*e. g.* Horiguchi 1981 [17], Aharony 1978 [3]) because two effects, frustration and dilution, can be investigated simultaneously. As before, (3.20) can be changed into a form of (3.11):

$$P(J_{ij}) = \mathrm{e}^{\alpha J_{ij}/J} \left\{ \delta(J_{ij} - J) + \delta(J_{ij}) + \delta(J_{ij} + J) \right\} \frac{\mathrm{e}^{\gamma (J_{ij}/J)^2}}{2\mathrm{e}^{\gamma} \cosh \alpha + 1} \tag{3.21}$$

with

$$\exp(-2\alpha) = \frac{q}{p} \quad \text{and} \quad \exp(\alpha + \gamma) = \frac{p}{r}. \tag{3.22}$$

Again it is easy to verify that (3.21) is equivalent to (3.20) by using (3.22). The condition for (3.10) to be valid is now $\alpha = \beta J$ or $\exp(-2\beta J) = q/p$. Since the present distribution (3.20) has one more parameter than the binary distribution does, the phase diagram is drawn in a three-dimensional space. The constraint $\exp(-2\beta J) = q/p$ therefore defines a surface in the phase diagram (see Figure 1 of Horiguchi, 1981 [17]). On the surface the internal energy is

$$\begin{aligned} \left[\, \langle E \rangle \,\right] &= -\frac{1}{2} zN \left[J_{ij} \right] \\ &= -\frac{1}{2} zNJ(p-q). \end{aligned}$$

3.2.5 Remark

The examples presented above may give an impression that the formula (3.10) is of no use because the internal energy (3.10) is not singular and it fails to provide information on phase transitions. Nevertheless we give here two excuses for devoting so much space to the calculation of the non-singular internal energy:

(i) Anyway the internal energy *is* obtained exactly for a general lattice. The result (3.10) would serve to check the validity of numerical calculations and full exact solutions in the future. (ii) An analogous way of considerations can be pursued in calculations of other physical quantities to give more useful results (see the following sections) .

3.3 Upper bound to the specific heat

An upper bound to the specific heat is obtained in this section. A condition of the finiteness of the specific heat is derived. As was in the previous section, only the spin 1/2 Ising model is treated.

3.3.1 General expression

The specific heat is the temperature derivative of the internal energy. It is equivalently said to be the average fluctuation of the internal energy:

$$kT^2\big[\,\langle C\rangle\,\big] = \big[\,\big\langle E^2\big\rangle\,\big] - \big[\,\langle E\rangle^2\,\big] \tag{3.23}$$

We demonstrate that the first term on the right-hand side of (3.23) can be calculated exactly if $A_{ij} = \beta J_{ij}$. However the second term may be evaluated only by an inequality.

The first term is rather easily calculated if $A_{ij} = \beta J_{ij}$. An expression corresponding to (3.7) can be given to the first term:

$$\big[\,\big\langle E^2\big\rangle\,\big] = \int_0^\infty \prod_{(i,j)} \mathrm{d}J_{ij} Q(J_{ij}) 2^{-N} \sum_\tau \sum_\sigma \exp\left(\sum_{(i,j)} \tau_{ij} A_{ij} \sigma_i \sigma_j\right) \frac{\dfrac{\partial^2}{\partial\beta^2} \displaystyle\sum_S \exp\left(\beta \sum_{(i,j)} J_{ij}\tau_{ij} S_i S_j\right)}{\displaystyle\sum_S \exp\left(\beta \sum_{(i,j)} J_{ij}\tau_{ij} S_i S_j\right)}.$$

As was in the calculation of the internal energy, the right-hand side of the above equation is simplified by the condition $A_{ij} = \beta J_{ij}$ into

$$\big[\,\big\langle E^2\big\rangle\,\big] = \int_0^\infty \prod_{(i,j)} \mathrm{d}J_{ij} Q(J_{ij}) 2^{-N} \sum_\tau \frac{\partial^2}{\partial\beta^2} \sum_S \exp\left(\beta \sum_{(i,j)} J_{ij}\tau_{ij} S_i S_j\right)\Bigg|_{A_{ij}=\beta J_{ij}}.$$

We perform the sum over $\{\tau_i\}$ first and then over $\{S_i\}$. Replacing $Q(J_{ij})$ by its definition, we find

$$\begin{aligned}
\big[\,\big\langle E^2\big\rangle\,\big] &= \frac{\partial^2}{\partial\beta^2}\left\{\int_0^\infty \mathrm{d}J_{ij}\,(P(J_{ij}) + P(-J_{ij}))\,\frac{\cosh\beta J_{ij}}{\cosh A_{ij}}\right\}^{zN/2}\Bigg|_{A_{ij}=\beta J_{ij}} \\
&= -\frac{1}{2}zN\left(\frac{1}{2}zN - 1\right)\left\{\int_0^\infty \mathrm{d}J_{ij}\,(P(J_{ij}) + P(-J_{ij}))\,J_{ij}\tanh\beta J_{ij}\right\}^2 \\
&\quad + \frac{1}{2}zN\int_{-\infty}^\infty \mathrm{d}J_{ij} P(J_{ij}) J_{ij}^2.
\end{aligned} \tag{3.24}$$

In this equation the integration including $\tanh\beta J_{ij}$ can be further simplified by the relation (3.9) to yield

$$\big[\,\big\langle E^2\big\rangle\,\big] = \frac{1}{2}zN\left(\frac{1}{2}zN - 1\right)\big[J_{ij}^2\big] + \frac{1}{2}zN\big[J_{ij}^2\big]. \tag{3.25}$$

The second term on the right-hand side of (3.23) is hard to be calculated exactly. The configurational average of $\langle E\rangle^2$ is written as

$$\left[\,\langle E\rangle^2\,\right] = \int_0^\infty \prod_{(i,j)} \mathrm{d}J_{ij} Q(J_{ij}) 2^{-N} \sum_\tau \sum_\sigma \exp\left(\sum_{(i,j)} \tau_{ij} A_{ij} \sigma_i \sigma_j\right) \times \left\{ \frac{\dfrac{\partial}{\partial\beta} \displaystyle\sum_S \exp\left(\beta \sum_{(i,j)} J_{ij}\tau_{ij}\sigma_i\sigma_j\right)}{\displaystyle\sum_S \exp\left(\beta \sum_{(i,j)} J_{ij}\tau_{ij}\sigma_i\sigma_j\right)} \right\}^2 . \tag{3.26}$$

Since the last factor is squared in (3.26), there remains a complicated quantity in the denominator even when $A_{ij} = \beta J_{ij}$. However, a trivial inequality $[\langle E\rangle^2] \geq [\langle E\rangle]^2$ helps us to evaluate a lower bound of $[\langle E\rangle^2]$ (or an upper bound of $[\langle C\rangle]$). Since $[\langle E\rangle]$ is already obtained in the previous section (see (3.9) which is valid when $A_{ij} = \beta J_{ij}$), an upper bound to the specific heat is derived in the subspace $A_{ij} = \beta J_{ij}$ as

$$\begin{aligned} kT^2\left[\,\langle C\rangle\,\right] &= \left[\,\left\langle E^2\right\rangle\,\right] - \left[\,\langle E\rangle^2\,\right] \\ &\leq \left[\,\left\langle E^2\right\rangle\,\right] - \left[\,\langle E\rangle\,\right]^2 \\ &= \frac{1}{2}zN\left(\frac{1}{2}zN - 1\right)\left[J_{ij}\right]^2 + \frac{1}{2}zN\left[J_{ij}^2\right] - \left\{\frac{1}{2}zN\left[J_{ij}\right]\right\}^2 \\ &= \frac{1}{2}zN\left\{\left[J_{ij}^2\right] - \left[J_{ij}\right]^2\right\}, \end{aligned} \tag{3.27}$$

where (3.25) was used. The last expression in (3.27) shows that the specific heat is bounded by the mean fluctuation of J_{ij} if $A_{ij} = \beta J_{ij}$. Ordinary distribution functions have finite standard deviations of J_{ij}, and therefore we conclude that the specific heat is finite in the subspace $A_{ij} = \beta J_{ij}$ of the phase diagram.

3.3.2 Examples

The first example is the Gaussian distribution (3.12). As was shown in the calculation of the internal energy, the condition $A_{ij} = \beta J_{ij}$ takes the form $\beta J^2 = J_0$ for the Gaussian distribution. On the curve $\beta J^2 = J_0$ in the phase diagram the specific heat satisfies, according to (3.27),

$$kT^2\left[\,\langle C\rangle\,\right] \leq \frac{1}{2}zNJ^2. \tag{3.28}$$

As is illustrated in Figure 3.1 the curve $\beta J^2 = J_0$ generally crosses phase boundaries. However the specific heat is never divergent at the intersections of the curve and phase boundaries as is evident in (3.28).

To compare with the solution of Sherrington and Kirkpatrick (1975 [37]), the long-range limit of interactions is taken by rescaling $J_0 = \tilde{J}_0/N$, $J = \tilde{J}/\sqrt{N}$ and $z = N-1 \sim N$.

On the line $\beta\tilde{J}^2 = \tilde{J}_0$, (3.28) shows that the specific heat is finite:

$$kT^2\big[\,\langle C\rangle\,\big] \leq \frac{1}{2}N\tilde{J}^2. \tag{3.29}$$

On the other hand, according to Sherrington and Kirkpatrick (1975 [37]), the specific heat is given by

$$kT^2\big[\,\langle C\rangle\,\big] = N\left\{m\frac{\partial m}{\partial\beta}\tilde{J}_0 + \frac{1}{2}\tilde{J}^2(1-q^2) - \beta\tilde{J}^2 q\frac{\partial q}{\partial\beta}\right\}.$$

In the paramagnetic phase both of the order parameters vanish $m = q = 0$, and therefore

$$kT^2\big[\,\langle C\rangle\,\big] = \frac{1}{2}zN\tilde{J}^2.$$

This expression of the specific heat concludes with the upper bound at the right-hand side of (3.29). Thus the equality holds in (3.29) in the paramagnetic state.

The second example is the diluted binary distribution (3.20). In the subspace $\exp(-2\beta J) = q/p$ (see section 3.2.4 for the derivation of this condition) the specific heat satisfies, by (3.27),

$$kT^2\big[\,\langle C\rangle\,\big] \leq \frac{1}{2}zNJ^2(4pq + pr + qr)$$

which is finite. The binary distribution (3.16) is a special case $r = 0$ of the diluted binary one. If $r = 0$,

$$kT^2\big[\,\langle C\rangle\,\big] \leq \frac{1}{2}zNJ^2 4p(1-p) = \frac{1}{2}zNJ^2\mathrm{sech}^2\beta J. \tag{3.30}$$

This result has also been obtained by Morita and Horiguchi (1980 [32]). As is seen in Figure 3.2, the curve $\exp(-2\beta J) = (1-p)/p$, on which we can prove (3.30), intersects phase boundaries. The specific heat is nevertheless finite and hence singularities of the specific heat, if any, should be weak ones.

3.4 Gauge invariant correlation functions

3.4.1 Definition and discussion

The gauge invariant correlation function

$$C_\Gamma(ij) = \left\langle S_i\left(\prod_\Gamma \tau\right)S_j\right\rangle \tag{3.31}$$

was introduced by Fradkin *et al* (1978 [12]). They asserted that gauge invariance is one of the most important principles in our investigation of random spin systems, for the randomness of the frustration-free Mattis model is not gauge invariant and is easily erased away. However we have to be careful not to take gauge invariance too seriously;

gauge invariance is just one of the *consequences* of our formulation of the problem. Gauge invariance is not a guiding principle in constructing a theory, in contrast to the theories of elementary particles. We should not say that a gauge invariant quantity is more fundamental than a gauge non-invariant one.

Of course the gauge invariant correlation function is sometimes useful in obtaining a physical picture on the phase transition in random systems (Fradkin *et al* 1978 [12]. Schuster 1979 [36]). But this usefulness never implies superiority of the gauge invariant correlation function to gauge non-invariant ones. Investigation of the latter is rather more important experimentally.

With all the above remarks in mind, we calculate a property of the gauge invariant correlation function of the random Ising model. We show that the configurational average of the gauge invariant correlation function decays like $\exp(-aL)$ $(a > 0)$ where L is the length of the path r. This result can be proved in the subspace $A_{ij} = \beta J_{ij}$. No qualitative change is observed even across phase boundaries. This conclusion is in accord with the Schuster's (1979 [36]) finding: He showed that the configurational average of the frustration function (which is a special case of the gauge invariant correlation function) has a trivial exponential decay if the distribution of J_{ij} is binary. Our result is thus a generalization of his statement to arbitrary gauge invariant correlation functions with an arbitrary distribution function. Physically our conclusion is that the gauge-invariant function does not make sense if the configurational average is taken.

3.4.2 Configurational average of the gauge invariant correlation function

We prove here that, if $A_{ij} = \beta J_{ij}$, the configurational average of the gauge invariant correlation function (3.31) decays exponentially as the length L of the path Γ increases. As before, we start from an expression of the configurational average corresponding to (3.7)

$$\langle C_\Gamma(ij)\rangle = \int_0^\infty \prod_{(i,j)} \mathrm{d}J_{ij} Q(J_{ij}) 2^{-N} \sum_\tau \sum_\sigma \exp\left(\sum_{(i,j)} \tau_{ij} A_{ij} \sigma_i \sigma_j\right) \times \frac{\prod_\Gamma \dfrac{\partial}{\partial(\beta J_\Gamma)} \sum_S \exp\left(\beta \sum J_{ij}\tau_{ij} S_i S_j\right)}{\sum_S \exp\left(\beta \sum J_{ij}\tau_{ij} S_i S_j\right)}, \tag{3.32}$$

where the differentiation by βJ_{kl} is carried out on all bonds along the path Γ. If $A_{ij} = \beta J_{ij}$ in (3.32), a simplification of the same kind as before happens:

$$[C_\Gamma(ij)] = \int_0^\infty \prod_{(i,j)} \mathrm{d}J_{ij} Q(ij) 2^{-N} \sum_\tau \prod_\Gamma \frac{\partial}{\partial(\beta J_\Gamma)} \sum_S \exp\left(\beta \sum_{(i,j)} J_{ij}\tau_{ij} S_i S_j\right)\Bigg|_{A_{ij}=\beta J_{ij}}.$$

By performing the sums over τ and S, and then differentiating by βJ_Γ, we easily find

$$[C_\Gamma(ij)] = \int_0^\infty \prod_{(i,j)} \mathrm{d}J_{ij} Q(J_{ij}) \prod_\Gamma 2\sinh\beta J_{lk} \prod_{(kl)\neq\Gamma} 2\cosh\beta J_{kl} \Bigg|_{A_{ij}=\beta J_{ij}}, \tag{3.33}$$

where the last factor is a product of $2\cosh\beta J_{kl}$ over all bonds on the lattice except those on the path Γ. Using the relation (3.9), we can further simplify (3.33)

$$\begin{aligned} [C_\Gamma(ij)] &= \left\{ \int_0^\infty \mathrm{d}J_{kl} (P(J_{kl}) + P(-J_{kl})) \tanh\beta J_{kl} \right\}^L \\ &= \left\{ \int_{-\infty}^\infty \mathrm{d}J_{kl} \mathrm{sgn}(J_{kl}) P(J_{kl}) \right\}^L . \end{aligned} \tag{3.34}$$

Since the integration in (3.34) evidently gives a value less then one, an exponential decay of $[C_\Gamma]$ is concluded:

$$[C_\Gamma(ij)] = \exp(-aL)$$

with

$$a = -\ln \int_{-\infty}^\infty \mathrm{d}J_{kl} \mathrm{sgn}(J_{kl}) P(J_{kl}) > 0.$$

Therefore the configurational average of the gauge invariant correlation function has been proved to decay exponentially even in the ferromagnetic phase, because the subspace $A_{ij} = \beta J_{ij}$ (on which (3.34) is valid) extends into the ferromagnetic region (see Figures 3.1 and 3.2).

3.5 Ordinary correlation functions

Ordinary (gauge non-invariant) correlation functions have direct physical significance. We have failed to evaluate exact values of correlation functions, but two rigorous relations on the spin 1/2 random Ising systems are presented here to help our understanding of their behavior.

3.5.1 Griffiths inequality

In the subspace $A_{ij} = \beta J_{ij}$, we can prove an inequality

$$\left[\left\langle S^X \right\rangle\right] \geq 0, \tag{3.35}$$

where S^X is an arbitrary product of spins in the system. Horiguchi (1981 [17]) proved the relation (3.35) for the diluted binary distribution and here we generalize his argument to an arbitrary distribution function. The relation (3.35) resembles the first Griffiths inequality (Griffiths 1967 [14], 1972 [15]) for ferromagnetic systems. Physically the inequality (3.35) simply means that in the subspace $A_{ij} = \beta J_{ij}$ ferromagnetic interactions are dominant.

For proof, since S^X is not gauge invariant, we have to start from a relation like (3.2) and see the effects of gauge transformation (3.4). The configurational average of the correlation function is, corresponding to (3.2),

$$[\langle S^X \rangle] = \int_0^\infty \prod_{(i,j)} \mathrm{d}J_{ij} Q(J_{ij}) \sum_\tau \exp\left(\sum_{(i,j)} \tau_{ij} A_{ij}\right) \frac{\sum_S S^X \exp\left(\beta \sum_{(i,j)} \tau_{ij} J_{ij} S_i S_j\right)}{\sum_S \exp\left(\beta \sum_{(i,j)} \tau_{ij} J_{ij} S_i S_j\right)}, \tag{3.36}$$

where A_{ij} is defined by (3.3). We now apply the gauge transformation (3.4) to (3.36), sum over all $\{\sigma\}$ and then divide by 2^N. The result is

$$\begin{aligned}
[\langle S^X \rangle] &= \int_0^\infty \prod_{(i,j)} \mathrm{d}J_{ij} Q(J_{ij}) 2^{-N} \sum_\tau \sum_\sigma \sigma^X \exp\left(\sum_{(i,j)} \tau_{ij} A_{ij} \sigma_i \sigma_j\right) \langle S^X \rangle \\
&= \int_0^\infty \prod_{(i,j)} \mathrm{d}J_{ij} Q(J_{ij}) 2^{-N} \sum_\tau \left\{ \sum_\sigma \sigma^X \exp\left(\sum_{(i,j)} \tau_{ij} A_{ij} \sigma_i \sigma_j\right) \right\} \langle \sigma^X \rangle \langle S^X \rangle ,
\end{aligned} \tag{3.37}$$

where $\langle \sigma^X \rangle$ is evaluated by the interaction $\tau_{ij} A_{ij} \sigma_i \sigma_j$ and $\langle S^X \rangle$ is by $\beta J_{ij} S_i S_j$. If $A_{ij} = \beta J_{ij}$, $\langle \sigma^X \rangle$ is equal to $\langle S^X \rangle$ since both S and σ are Ising variables. Therefore $[\langle S^X \rangle] \geq 0$ 0 if $A_{ij} = \beta J_{ij}$ (note that all quantities but $\langle \sigma^X \rangle \langle S^X \rangle$ are positive definite in (3.37)).

3.5.2 Average of the inverse correlation function

The next relation to be proved is

$$\left[\frac{1}{\langle S^X \rangle}\right] = 1, \tag{3.38}$$

where S^X denotes a product of even number of spin variables. Equation (3.38) is valid only if $A_{ij} = \beta J_{ij}$. Our proof is analogous to that of the Griffiths inequality in the previous subsection. The configurational average is, corresponding to (3.36),

$$\left[\frac{1}{\langle S^X \rangle}\right] = \int_0^\infty \prod_{(i,j)} \mathrm{d}J_{ij} Q(J_{ij}) \sum_\tau \exp\left(\sum_{(i,j)} \tau_{ij} A_{ij}\right) \frac{\sum_S \exp\left(\beta \sum_{(i,j)} \tau_{ij} J_{ij} S_i S_j\right)}{\sum_S S^X \exp\left(\beta \sum_{(i,j)} \tau_{ij} J_{ij} S_i S_j\right)}, \tag{3.39}$$

where it is understood that S^X should be a product of *even* number of spins because $\langle S^X \rangle$ would be vanishing otherwise. As before, we perform the gauge transformation

(3.4), summation over $\{\sigma\}$ and division by 2^N successively in (3.39). By making use of the relation $\sigma^X = 1/\sigma^X$ (since $(\sigma^X)^2 = 1$), we finally find

$$\left[\frac{1}{\langle S^X \rangle}\right] = \int_0^\infty \prod_{(i,j)} \mathrm{d}J_{ij} Q(J_{ij}) \sum_\tau \left\{ \sum_\sigma \sigma^X \exp\left(\sum_{(i,j)} \tau_{ij} A_{ij} \sigma_i \sigma_j \right) \right\} \times \frac{\sum_S \exp\left(\beta \sum_{(i,j)} \tau_{ij} J_{ij} S_i S_j \right)}{\sum_S S^X \exp\left(\beta \sum_{(i,j)} \tau_{ij} J_{ij} S_i S_j \right)}.$$

If A_{ij} is equal to βJ_{ij}, the complicated factors in the above equation (those including σ_X and S_X) cancel out, and other factors are easily evaluated to yield unity. Thus (3.38) has been proved.

We should remark that (3.38) does not imply absence of phase boundaries across the subspace $A_{ij} = \beta J_{ij}$. The situation is quite similar to that of the internal energy.

3.6 Refinement of the Horiguchi and Morita's inequality

A useful inequality was obtained by Horiguchi and Morita (1981 [19]) for classical random spin systems. We discuss in this section a refinement of their result.

3.6.1 Preliminaries

An inequality proved by Horiguchi and Morita (1981[19]) provides an excellent estimation of an upper bound to the ordinary (gauge non-invariant) correlation functions. The inequality proves that the spin glass phase may exist in a finite region in the phase diagram by showing that the ferromagnetic long range order vanishes in a finite subspace of the phase diagram. We demonstrate that a reinterpretation of their proof of the inequality gives a refined upper bound. The refined relation yields far more information on the phase diagram than one expects from its simplicity.

In the present and the following sections we consider a random spin system with an arbitrary spin symmetry (Ising classical Heisenberg and so on) in external magnetic field h. Since we are particularly interested in the spontaneous magnetization (ferromagnetic long-range order), we should be careful about the treatment of the thermodynamic limit and the zero-field limit.

Suppose that the system is on a finite set of sites Λ:

$$H = -\sum_{i,j \in \Lambda} J_{ij} S_i S_j - h \sum_{i \in \Lambda} S_i. \tag{3.40}$$

Following Horiguchi and Morita (1981 [19]) ,we estimate an upper bound to the spin correlation function

$$\lim_{h\to+0}\lim_{N\to\infty}\left|\left[\langle S_A^\rho\rangle_{N,h,B_0}^{\{\beta J_{ij}\}}\right]\right| \equiv \left|\left[\langle S_A^\rho\rangle^{\{\beta J_{ij}\}}\right]\right|, \tag{3.41}$$

where

$$S_A = \prod_{k\in A\subset\Lambda}(S_k)^{\rho(k)} \qquad (\rho(k)\text{: integer} \geq 0) \tag{3.42}$$

is a product of all spins (and their powers) in a subspace A of Λ. In (3.41) N is the total number of sites in Λ, and B_0 denotes the boundary condition (free boundary as implicit in (3.40)). As will be shown later, two upper bounds to (3.41) are written in terms of correlation functions of the following "reference systems":

$$H_1 = -\sum_{i,j\in\Lambda_1} J\sigma_i\sigma_j - \sum_{i\in\partial\Lambda_1} J\sigma_i \quad (\sigma_i = \pm 1) \tag{3.43}$$

and

$$H_2 = -\sum_{i,j\in\Lambda_1}\gamma_{ij}\sigma_i\sigma_j - \sum_{i\in\Lambda_1, j\in\Lambda\setminus\Lambda_1}\gamma_{ij}\sigma_i - \sum_{i,j\in\Lambda\setminus\Lambda_1}\gamma_{ij} \quad (\sigma_i = \pm 1) \tag{3.44}$$

where Λ_1 is a subset of Λ including A ($A \subset \Lambda_1 \subset \Lambda$), and γ_{ij} is a quenched random variable to be defined later precisely. In these systems, (3.43) and (3.44), we may consider correlation functions similar to (3.41),

$$\lim_{N_1\to\infty}\langle\sigma_A^\rho\rangle_{N_1,B_1}^{\{\beta J\}} \equiv \langle\sigma_A^\rho\rangle^{(\beta J)} \tag{3.45}$$

for H_1 and

$$\lim_{N_1\to\infty}\left[\left|\langle\sigma_A^\rho\rangle_{N_1,B_1}^{\{\beta\gamma_{ij}\}}\right|\right] \equiv \left[\left|\langle\sigma_A^\rho\rangle^{\{\beta\gamma_{ij}\}}\right|\right] \tag{3.46}$$

for H_2, where N_1 is the number of sites in Λ_1, B_1 denotes the boundary condition of H_2 ($\sigma_i = 1$ outside Λ_1), and σ_A^ρ is defined in the same way as (3.42).

3.6.2 Horiguchi and Morita's inequality and a refinement

The correlation inequality to be proved are the following ones:

$$\left|\left[\langle S_A^\rho\rangle^{\{\beta J_{ij}\}}\right]\right| \leq |S|^{|A|}\left|\left[\langle\sigma_A^\rho\rangle^{\{A_{ij}\}}\right]\right| \tag{3.47}$$

$$\leq |S|^{|A|}\langle\sigma_A^\rho\rangle^{([A_{ij}])}, \tag{3.48}$$

where $|S|$ is the maximum absolute value of S_i, $|A|$ is defined by

$$|A| = \sum_{k\in A}\rho(k) \tag{3.49}$$

and A_{ij} is the same quantity as before, Eqn. (3.3)[2]. In (3.48) the correlation function is evaluated by the Hamiltonian H_1 ((3.43), see also (3.45)) with βJ replaced by the configurational average of A_{ij}. Horiguchi and Morita proved the inequality (3.48) (compared to the left-hand side of (3.47)). An inspection of their proof, however, reveals that the relation (3.47) is valid and we demonstrate that (3.47) for an infinite system is informative on the shape of phase boundaries.

3.6.3 Proof

It is convenient to introduce another system where the external field is applied only outside Λ_1:

$$H_3 = -\sum_{i,j\in\Lambda} J_{ij}S_iS_j - h\sum_{i\in\Lambda\setminus\Lambda_1} S_i. \tag{3.50}$$

The spin correlation function of our interest is, corresponding to (3.41),

$$\left[\langle S_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}\right] = \int_0^\infty \prod \mathrm{d}J_{ij}Q(J_{ij}) \sum_\tau \exp\left(\sum_{i,j\in\Lambda}\tau_{ij}A_{ij}\right)\langle S_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}, \tag{3.51}$$

where $(h,0)$ denotes the way that the field is applied in H_3 ($h=0$ inside Λ_1). We perform a gauge transformation here. The variables $\{\sigma_i\}$ of the gauge transformation, as defined by (3.4), are now chosen to be $+1$ outside Λ_1 and arbitrarily fixed to $+1$ or -1 inside Λ_1. Since the field h is applied only outside Λ_1 in the system H_3, this gauge transformation leaves the Hamiltonian H_3 invariant. Consequently the correlation function (3.51) is transformed into

$$\left[\langle S_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}\right] = \int_0^\infty \prod \mathrm{d}J_{ij}Q(J_{ij}) \sum_\tau \exp\left(\sum_{i,j\in\Lambda}\tau_{ij}A_{ij}\sigma_i\sigma_j\right)\sigma_A^\rho\,\langle S_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}, \tag{3.52}$$

where σ_i with i outside Λ_1 should be read as $+1$. Since (3.52) is valid for any choice of $\{\sigma_i\}$ ($i\in\Lambda_1$), we may sum over $\{\sigma_i\}$ ($i\in\Lambda_1$) and then divide by 2^{N_1}. These manipulations do not affect the value of (3.52). Only the representation is changed into

$$\begin{aligned}\left[\langle S_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}\right] &= \int_0^\infty \prod \mathrm{d}J_{ij}Q(J_{ij}) \sum_\tau 2^{-N_1} \\ &\times \sum_{\sigma\in\Lambda_1}\sigma_A^\rho \exp\left(\sum_{i,j\in\Lambda_1}\tau_{ij}A_{ij}\sigma_i\sigma_j + \sum_{i\in\Lambda_1,j\in\Lambda\setminus\Lambda_1}\tau_{ij}A_{ij}\sigma_i + \sum_{i,j\in\Lambda\setminus\Lambda_1}\tau_{ij}A_{ij}\right)\langle S_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}},\end{aligned} \tag{3.53}$$

[2]We notice that the similarity of the notation A (for the set) to A_{ij} (for the effective interaction) never implies a close relationship between them.

where $\sigma_i = 1$ (outside Λ_1) is explicitly taken into account. This relation may be rewritten as

$$\left[\langle S_A^\rho \rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}\right] = \int_0^\infty \prod \mathrm{d}J_{ij} Q(J_{ij}) \sum_\tau 2^{-N_1}$$
$$\times \sum_{\sigma\in\Lambda_1} \exp\left(\sum_{i,j\in\Lambda_1} \tau_{ij}A_{ij}\sigma_i\sigma_j + \sum_{i\in\Lambda_1, j\in\Lambda\setminus\Lambda_1} \tau_{ij}A_{ij}\sigma_i + \sum_{i,j\in\Lambda\setminus\Lambda_1} \tau_{ij}A_{ij}\right) \langle\sigma_A^\rho\rangle_{N_1,B_1}^{\{A_{ij}\}} \langle S_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}, \tag{3.54}$$

where, as in (3.45) and (3.46), B_1 denotes the boundary condition ($\sigma_1 = +1$ outside Λ_1). We now estimate an upper bound to the absolute value of (3.54). Note that

$$\left|\langle S_A^\rho \rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}\right| \le |S|^{|A|}$$

with $|A|$ defined by (3.49). The correlation (3.54) is thus bounded as follows:

$$\left|\left[\langle S_A^\rho \rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}\right]\right| \le |S|^{|A|} \int_0^\infty \prod \mathrm{d}J_{ij} Q(J_{ij}) \sum_\tau 2^{-N_1}$$
$$\times \sum_{\sigma\in\Lambda_1} \exp\left(\sum_{i,j\in\Lambda_1} \tau_{ij}A_{ij}\sigma_i\sigma_j + \sum_{i\in\Lambda_1, j\in\Lambda\setminus\Lambda_1} \tau_{ij}A_{ij}\sigma_i + \sum_{i,j\in\Lambda\setminus\Lambda_1} \tau_{ij}A_{ij}\right) \left|\langle\sigma_A^\rho\rangle_{N_1,B_1}^{\{A_{ij}\}}\right|. \tag{3.55}$$

This inequality is equivalent to

$$\left|\left[\langle S_A^\rho \rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}\right]\right| \le |S|^{|A|} \int_0^\infty \prod \mathrm{d}J_{ij} Q(J_{ij}) \sum_\tau \exp\left(\sum_{i,j\in\Lambda} \tau_{ij}A_{ij}\right) \left|\langle\sigma_A^\rho\rangle_{N_1,B_1}^{\{A_{ij}\}}\right|, \tag{3.56}$$

because (3.56) can be transformed back into (3.55) by the same manipulations as those which led us from (3.51) to (3.53). Equation (3.56) indicates that am upper bound is given by the configurational average of the absolute value of the correlation function $\langle\sigma_A^\rho\rangle$:

$$\left|\left[\langle S_A^\rho \rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}\right]\right| \le |S|^{|A|} \left[\left|\langle\sigma_A^\rho\rangle_{N,B_1}^{\{A_{ij}\}}\right|\right]. \tag{3.57}$$

Horiguchi and Morita (1979 [18]) proved that the configurational average on the right-hand side of (3.57) is bounded by a correlation function of a non-random system:

$$\begin{aligned}\left|\left[\langle S_A^\rho \rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}\right]\right| &\le |S|^{|A|} \left[\left|\langle\sigma_A^\rho\rangle_{N,B_1}^{\{A_{ij}\}}\right|\right] \\ &\le |S|^{|A|} \langle\sigma_A^\rho\rangle_{N,B_1}^{([A_{ij}])},\end{aligned} \tag{3.58}$$

where the superscript $([A_{ij}])$ means that the thermal average is carried out on a pure (non-random) system with the effective interaction βJ replaced by the configurational average of A_{ij}, see (3.45). We take here the thermodynamic limit $N \to \infty$ and zero-field

limit $h \to +0$. According to Horiguchi and Morita (1981 [19]) the expectation value of S_A^ρ in these limits becomes independent of the choice of the way that field is applied:

$$\begin{aligned} \lim_{h\to+0}\lim_{N\to\infty}\left|\left[\langle S_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}\right]\right| &= \lim_{h\to+0}\lim_{N\to\infty}\left|\left[\langle S_A^\rho\rangle_{N,h,B_0}^{\{\beta J_{ij}\}}\right]\right| \\ &\equiv \left|\left[\langle S_A^\rho\rangle^{\{\beta J_{ij}\}}\right]\right|. \end{aligned}$$

The expectation values of σ_A^ρ in (3.58) are not affected by the limits $N \to \infty$ and $h \to +0$; these expectation values of σ_A^ρ depend only on the variable inside Λ_1 without applied field. We should take another thermodynamic limit $N_1 \to \infty$ separately. From the definitions (3.45) and (3.46) it is clear that the limit $N_1 \to \infty$ in (3.58) yields the relations (3.47) and (3.48). This completes the proof.

A comment is made here on the existence of the limit in (3.46). Since the existence of the thermodynamic limit of correlation functions in random spin systems is not yet established well enough, we may not assert that the inequality (3.47) is rigorously proved. However, in our applications of (3.47) we consider only the case that (3.46) actually converges to zero. Therefore the following applications of (3.47) are perfectly legitimate although (3.47) itself cannot be said to be proved to always have a definite sense in the thermodynamic limit.

3.6.4 Restrictions on phase diagrams

To illustrate the usefulness of the refined Horiguchi and Morita's inequality (3.47) in drawing phase diagrams, we consider a random Ising model ($S_i = \pm 1$ in (3.40)) with the binary distribution function (3.16). Let the product of spins S_A^ρ be simply a single-site spin S_0. Then we may write the left-hand side of (3.47), as $m(\beta J, p)$ because it is the spontaneous magnetization, see (3.41), evaluated at the interaction strength $\pm\beta J$ with p being the probability of the positive interaction. When the spin variable is of the Ising type, $|S| = 1$. In the right-hand side of (3.47) the effective interaction A_{ij} is now K_p defined by (3.18) since the distribution is binary. Thus the right-hand side of (3.47), compared with (3.46), may be written as $g(K_p, p)$ (with the implication that the two-variable function $g(\beta J, p)$ is evaluated, in (3.47), on the line $\beta J = K_p$ – which is equal to the line $\exp(-2\beta J) = (1-p)/p$ often appeared in previous sections). With these notations, the inequality (3.47) is represented as

$$0 \le |m(\beta J, p)| \le g(K_p, p). \tag{3.59}$$

The above relation may be interpreted as follows. In the paramagnetic phase the expectation value of a single site spin vanishes in the thermodynamic limit for any boundary conditions. Since the function g in (3.59) is the expectation value of σ_0, $g(K_p, p)$ vanishes when the line $\beta J = K_p$ (or $\exp(-2\beta J) = (1-p)/p$) is in the paramagnetic phase. Equation (3.59) is thus understood to provide a criterion to determine whether a point $B(\beta J, p)$ in the phase diagram is in the ferromagnetic region or not; if the line $\beta J = K_p$ is in the paramagnetic phase at the same p as the point B, then $m = 0$ at B irrespective of βJ. This criterion is more conveniently restated in the following way. If the line $\beta J = K_p$ intersects the phase boundary between the paramagnetic and ordered (ferromagnetic or

spin glass) phases at $p = p_0$, then the line $\beta J = K_p$ lies in the paramagnetic phase for $1/2 < p < p_0$ by topological considerations (see Figure 3.2). It readily follows that the spontaneous magnetization is absent for $1/2 < p < p_0$. A few interesting examples are given below.

Let us suppose that the spin glass phase does not exist at finite temperatures. The random Ising model on the square lattice is expected to have this property (Morgenstern and Binder, 1979 [31]). If we denote the boundary curve (between the ferromagnetic and paramagnetic phases) by $T = T(p)$, it can be proved that $\mathrm{d}T(p)/\mathrm{d}p = \infty$ at $p = p_0$ (Figures 3.3(a), 3.3(b)) or the boundary $T = T(p)$ is non-differentiable at $p = p_0$ (Figure 3.3(c)); otherwise (if $\mathrm{d}T(p)/\mathrm{d}p$ exists at $p = p_0$ and is finite there) the ferromagnetic long range order m becomes non-vanishing for $p < p_0$ (Figures 3.3(d) and 3.3(e)), a contradiction. If $\mathrm{d}T(p)/\mathrm{d}p$ diverges at p_0, two possibilities exist on the shape of the phase boundary. First, the curve $T = T(p)$ itself may have no singularities (Figure. 3.3(a)). In this case, for p fixed slightly greater than p_0, the system experiences two phase transitions, as the temperature is lowered from infinity, from $m = 0$ to $m \neq 0$ and then to $m = 0$ again. This means that the paramagnetic state is more stable than the ferromagnetic one at low temperatures. The second possibility is illustrated in Figure. 3.3(b); the phase boundary may have a singular point at ($p = p_0$, $T = T_0$) below which ($T < T_0$) the boundary is a vertical line. If this is the case, the paramagnetic phase is no more stable than the ferromagnetic one at low temperatures.

If the spin glass phase exists, (3.59) provides restrictions on the shape of the phase diagram in a somewhat different manner. First, if the line $\beta J = K_p$ intersects the phase boundary of the ferromagnetic and paramagnetic phases, the same argument as before is applied to prove the vertical slope (or non-differentiability) of the boundary curve at the intersection of the boundary and the line $\beta J = K_p$. In this case the boundary between the ferromagnetic and spin glass phases should also lie in the region $p > p_0$ because $m = 0$ when $p \geq p_0$ (Figure 3.4).

If the curve $\beta J = K_p$ intersects the boundary of the spin glass and paramagnetic phases at $p = p_0$, Figure 3.5, then the ferromagnetic phase cannot exist for $1/2 < p < p_0$. A slightly stronger restriction may be given in this case. The function $g(K_p, p)$ in (3.59) is defined as the thermodynamic limit of the expectation value of σ_0 under the boundary condition B_1. Under the condition B_1, all spins surrounding the system are fixed to up states, which may correspond to applying an infinitesimal external field on the free boundary system (see Griffiths, 1972 [15]). On the other hand a uniform external field may not be the relevant ordering field of the spin glass state (Suzuki, 1981 [39]). If this is the case, $g(K_p, p)$ vanishes even in the spin glass phase because g is not the correct order parameter of the spin glass phase. Consequently we can apply the relation (3.59) up to the phase boundary between the spin glass and ferromagnetic phases. The same argument as before applies to the spin glass-ferromagnetic boundary and it is concluded that this boundary curve should be locally vertical or non-differentiable at the intersection of the curve $\beta J = K_p$ and the phase boundary.

The last possibility is that the tricritical point (where the ferromagnetic, paramagnetic and spin glass phases meet) is on the line $\beta J = K_p$, see Figure 3.6 . If this is the case, we can predict that the boundary curve $T = T(p)$ between the ferromagnetic and spin glass phases has a negative slope $\mathrm{d}T(p)/\mathrm{d}p < 0$ as is evident in Figure 3.6.

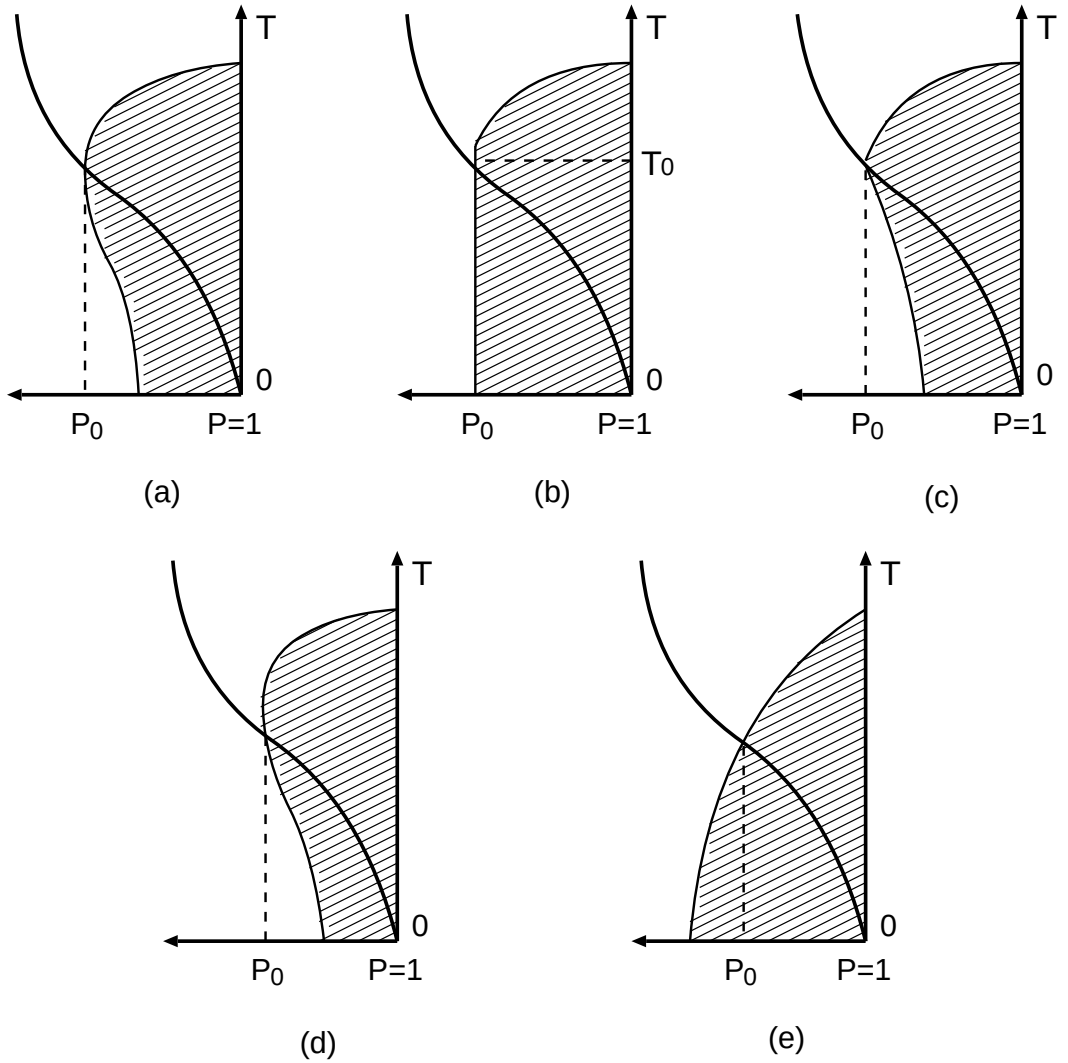

Figure 3.3:

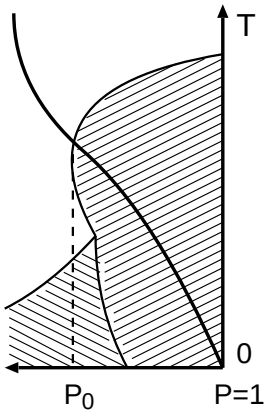

Figure 3.4:

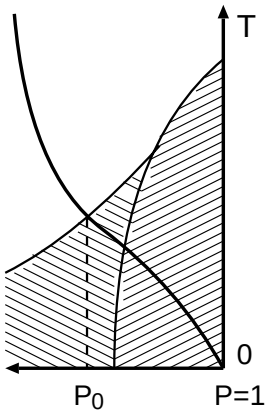

Figure 3.5:

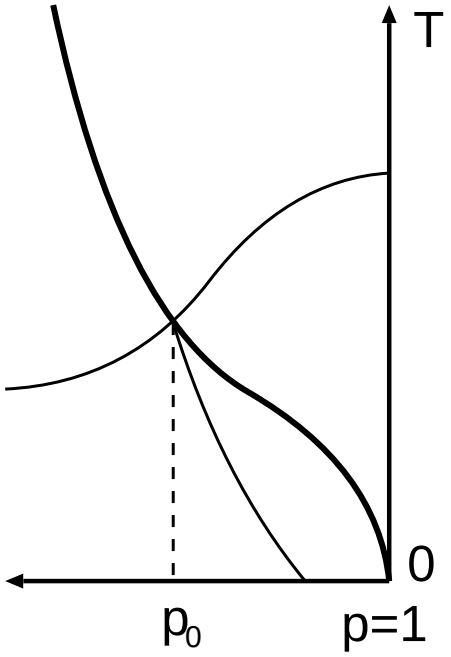

Figure 3.6:

Similar arguments apply to other distribution functions (*e. g.* the Gaussian) to yield analogous results.

3.6.5 Supplementary comments

In the previous subsection we confined ourselves to the spin 1/2 Ising model ($S_i = \pm 1$). Since the inequality (3.47) was obtained for an arbitrary classical spin S_i, one may wish to apply the relation (3.59) to non-Ising classical spin systems. However, the variables $\{\sigma_i\}$ on the right-hand side of (3.47) are Ising variables (because they are introduced by the gauge transformation). Accordingly in (3.59), m is the spontaneous magnetization of the spin system $\{S_i\}$ (*e. g.* the classical Heisenberg model) while g is defined for the Ising model on the same lattice. It follows that the region of absence of magnetization is restricted by the phase diagram of the random Ising model on the same lattice. We do not have rigorous phase diagrams of random Ising models (except those on special lattices, see below) and at present the inequality (3.59) is of no direct use to draw phase diagrams of non-Ising systems.

On some special lattices, such as the Bethe lattice (Katsura and Fujiki, 1979 [21]), the cactus tree (Ono, 1976 [34]) and so forth, rigorous phase diagrams are available. However all of these special lattices are infinite from the outset, and this fact enables us to solve them. Because of the infinite nature of these systems, our way of formulating and investigating the problem is not allowed to be applied to them.

3.7 Absence of spontaneous magnetization

In the previous section the spontaneous magnetization was proved to be absent in a finite region of the phase diagram. Another proof of the absence of spontaneous magnetization is presented in this section. The proof here has a more limited range of applicability than that in section 3.6 does; it is shown that $m = 0$ only in the *neighborhood* of the symmetric distribution function $P(J_{ij}) = P(-J_{ij})$. We cannot tell whether the extension of this "neighborhood" is finite or infinitesimal. Therefore the conclusion in the present section is weaker than that given by the Horiguchi and Morita's inequality. Nevertheless our proof here reveals an interesting physical mechanism of the absence of spontaneous magnetization, the locality of gauge transformation, while the proof of the Horiguchi and Morita's inequality was rather mathematical and gave little physical insight into the reason for the absence of magnetization.

We first show that $m = 0$ for an exactly symmetric distribution $P(J_{ij}) = P(-J_{ij})$. Then we argue that each term of the expansion of the magnetization from the symmetric distribution vanishes. This completes the proof of m=0 in the *neighborhood* of the symmetric distribution.

If the distribution is symmetric, the expectation value of the magnetization is

$$[\langle S_0^z \rangle_h] = 2^{-zN/2} \int_0^\infty \prod \mathrm{d}J_{ij} P(J_{ij}) \sum_\tau \frac{\mathrm{Tr} S_0^z \exp\left(\beta \sum J_{ij}\tau_{ij} S_i S_j + \beta h \sum S_i^z\right)}{\mathrm{Tr} \exp\left(\beta \sum J_{ij}\tau_{ij} S_i S_j + \beta h \sum S_i^z\right)}, \quad (3.60)$$

where S_i is a classical spin variable, 0 is an arbitrarily chosen site and the external field h is applied along the z-axis of the spin space. Let us perform a gauge transformation only at the site 0:

$$S_0 \to -S_0, \quad J_{0j} \to -J_{0j} \quad (j \text{ adjacent to } 0).$$

Equation (3.60) is transformed into

$$[\langle S_0^z \rangle_h] = -2^{-zN/2} \int_0^\infty \prod \mathrm{d}J_{ij} P(J_{ij}) \sum_\tau \frac{\mathrm{Tr} S_0^z \exp\left(\beta \sum J_{ij}\tau_{ij} S_i S_j + \beta h \sum S_i^z - 2\beta h S_0^z\right)}{\mathrm{Tr} \exp\left(\beta \sum J_{ij}\tau_{ij} S_i S_j + \beta h \sum S_i^z - 2\beta h S_0^z\right)}. \quad (3.61)$$

We insert now the partition function (denominator in (3.60)) into the numerator and denominator of the last factor in (3.61):

$$\begin{aligned}[\langle S_0^z \rangle_h] = & -2^{-zN/2} \int_0^\infty \prod \mathrm{d}J_{ij} P(J_{ij}) \sum_\tau \frac{\mathrm{Tr} \exp\left(\beta \sum J_{ij}\tau_{ij} S_i S_j + \beta h \sum S_i^z\right)}{\mathrm{Tr} \exp\left(\beta \sum J_{ij}\tau_{ij} S_i S_j + \beta h \sum S_i^z - 2\beta h S_0^z\right)} \\ & \times \frac{\mathrm{Tr} S_0^z \exp\left(\beta \sum J_{ij}\tau_{ij} S_i S_j + \beta h \sum S_i^z - 2\beta h S_0^z\right)}{\mathrm{Tr} \exp\left(\beta \sum J_{ij}\tau_{ij} S_i S_j + \beta h \sum S_i^z\right)}. \end{aligned} \quad (3.62)$$

We further rewrite by using the identify

$$\exp(\pm 2\beta h S_0^z) = 1 + \{\exp(\pm 2\beta h S_0^z) - 1\},$$

by which (3.62) is changed into

$$
\begin{aligned}
&[\langle S_0^z\rangle_h] \\
&= -2^{-zN/2}\int_0^\infty \prod \mathrm{d}J_{ij}P(J_{ij})\sum_\tau \frac{\mathrm{Tr}\{1+(\mathrm{e}^{2\beta hS_0^z}-1)\}\exp\left(\beta\sum J_{ij}\tau_{ij}S_iS_j+\beta h\sum S_i^z-2\beta hS_0^z\right)}{\mathrm{Tr}\exp\left(\beta\sum J_{ij}\tau_{ij}S_iS_j+\beta h\sum S_i^z-2\beta hS_0^z\right)} \\
&\quad\times\frac{\mathrm{Tr}S_0^z\{1+(\mathrm{e}^{-2\beta hS_0^z}-1)\}\exp\left(\beta\sum J_{ij}\tau_{ij}S_iS_j+\beta h\sum S_i^z\right)}{\mathrm{Tr}\exp\left(\beta\sum J_{ij}\tau_{ij}S_iS_j+\beta h\sum S_i^z\right)}. \qquad (3.63)
\end{aligned}
$$

Since the factors between $2^{-zN/2}$ and $\sum_\tau$ in (3.63) give the configurational average, we may write (3.63) as

$$
[\langle S_0^z\rangle_h] = -\left[\left\langle 1+(\mathrm{e}^{2\beta hS_0^z}-1)\right\rangle_{-h\text{ at }0}\left\langle S_0^z+S_0^z(\mathrm{e}^{-2\beta hS_0^z}-1)\right\rangle_h\right], \qquad (3.64)
$$

where "$-h$ at 0" implies that the thermal average is performed with the magnetic field h uniform but site 0. The field at site 0 is $-h$. Expansion of the right-hand side of (3.64) yields

$$
\begin{aligned}
[\langle S_0^z\rangle_h] &= -[\langle S_0^z\rangle_h]-\left[\left\langle S_0^z(\mathrm{e}^{-2\beta hS_0^z}-1)\right\rangle_h\right] \\
&\quad-\left[\left\langle \mathrm{e}^{2\beta hS_0^z}-1\right\rangle_{-h\text{ at }0}\langle S_0^z\rangle_h\right]-\left[\left\langle \mathrm{e}^{2\beta hS_0^z}-1\right\rangle_{-h\text{ at }0}\left\langle S_0^z(\mathrm{e}^{-2\beta hS_0^z}-1)\right\rangle_h\right]. \qquad (3.65)
\end{aligned}
$$

Now we take the thermodynamic limit. No change of expression is required in (3.65) in this limit. In the derivation of (3.62), we had to change the signs of the variables only at site 0 to leave the interaction $J_{ij}\tau_{ij}S_iS_j$ invariant. This locality of the transformation is directly reflected to the finiteness of the number of terms in (3.65). If the symmetry is global, for instance if the interaction is JS_iS_j, we should change the signs of *all* spins simultaneously to leave the interaction invariant. Then an extensive number of terms would appear in a relation corresponding to (3.65). The thermodynamic limit for such an equation would make the equation practically useless. In our present case, (3.65) has finite number of terms even in the thermodynamic limit. Next we observe that all terms but the first one on the right-hand side of (3.65) vanish in the zero-field limit. For instance, the second term may be estimated as follows:

$$
\begin{aligned}
\left|\left[\left\langle S_0^z(\mathrm{e}^{-2\beta hS_0^z}-1)\right\rangle_h\right]\right| &\le |S_0^z|_{\max}|\mathrm{e}^{-2\beta hS_0^z}-1|_{\max} \\
&\le |S_0^z|_{\max}|\mathrm{e}^{2\beta h|S_0^z|_{\max}}-1| \\
&\to 0 \quad \text{as } h\to 0\ .
\end{aligned}
$$

The third and fourth terms on the right-hand side of (3.65) can be shown to vanish in the same way. There for we obtain

$$
\left[\langle S_0^z\rangle_{h\to+0}\right] = -\left[\langle S_0^z\rangle_{h\to+0}\right] = 0.
$$

We next suppose that the distribution is a general (asymmetric) one. The configurational average of the magnetization is, corresponding to (3.2),

$$
[\langle S_0^z\rangle_h] = \int_0^\infty \prod \mathrm{d}J_{ij}Q(J_{ij})\sum_\tau \exp(\sum \tau_{ij}A_{ij})\,\langle S_0^z\rangle_h\,. \qquad (3.66)
$$

If the distribution is nearly symmetric, A_{ij} is almost zero as seen in (3.3). So we may expand the integrand of (3.66) in powers of $\tanh A_{ij}$ using the identify

$$\exp(\tau_{ij}A_{ij}) = \cosh A_{ij}(1 + \tau_{ij}\tanh A_{ij}).$$

A term of the expansion has an expression

$$\int_0^\infty \prod \mathrm{d}J_{ij}Q(J_{ij}) \sum_\tau \{\langle S_0^z\rangle_h \tau_{kl}\tau_{mn}\cdots\tanh A_{kl}\tanh A_{mn}\cdots\}$$
$$= [\langle S_0^z\rangle_h \tau_{kl}\tau_{mn}\cdots\tanh A_{kl}\tanh A_{mn}\cdots], \tag{3.67}$$

Where the configurational average $[\cdots]$ is evaluated at the symmetric distribution. We can prove that any quantity of the form (3.67) vanishes when $h \to +0$ by an analogous argument to the one for the symmetric case. The appropriate gauge transformation to obtain a relation similar to (3.64) is given as follows. (i) If the set of the sites $(klmn\cdots)$ in (3.67) includes the site 0 even times, we perform a gauge transformation at site 0. Then (3.67) acquires a minus sign and is proved to vanish. (ii) Otherwise, choose a site $j(\neq 0)$ which is included odd times in $(klmn\cdots)$. Such a site j always exists, since the site 0 is included odd times in the set $(klmn\cdots)$ and the set includes an even number of sites. The gauge transformation is performed at site j now, which gives a minus sign to (3.67). This completes the proof.

As has been emphasized above , the locality of the gauge transformation is the essential ingredient of the present proof. In this sense the absence of spontaneous magnetization bears a resemblance to the Elitzur's theorem (Elitzur 1975 [9]. Kogut 1979 [22]) of the lattice gauge theory. The Elitzur's theorem states that a spontaneous breakdown of a *local* symmetry is impossible.

3.8 Existence of ferromagnetic phase transition

In this section we prove the existence of the ferromagnetic phase in a finite region near the purely ferromagnetic limit $P(J_{ij}) \sim \delta(J_{ij} - J)$ in the phase diagram of the random Ising model. We do not make use of gauge transformations in the proof, but the result is found to be in close relation with the conclusions of previous sections. In particular the subspace defined by $a = \beta$ in (3.11) is again shown to play a crucial role in our argument.

To show the existence of the ferromagnetic long-range order we apply the Peierls (1936 [35]) argument made rigorous by Griffiths (1964 [13], 1972 [15]). The lattice structure is assumed to be a regular one. We focus our attention to the spin l/2 random Ising model although a generalization to higher spin Ising systems may be possible (see Lebowitz and Gallavotti, 1971 [23]). The essential idea of the Peierls argument is best illustrated on the square lattice, and so we treat this lattice first. Generalization to higher dimensions is commented on later. To avoid unimportant complications we consider only the binary distribution (3.16) and the Gaussian distribution (3.12) explicitly. Generalization to other distributions seems to be quite straightforward.

Our statement to be proved is as follows: The random Ising model on the square lattice has non-vanishing spontaneous magnetization at low temperatures if the distribution of J_{ij} is binary and $K_p > \beta J$ (see (3.18)) or if the distribution is Gaussian and $J_0 > \beta J^2$.

Let us prove this theorem. Consider a finite domain Λ on the square lattice. The boundary of the domain is assumed to have a regular shape (so that the energy of the boundary can be neglected in the thermodynamic limit). All spins on the boundary are in the up state ($S_i = +1$). As will be explained precisely later, this boundary condition is, in a sense, equivalent to an infinitesimal external field ($h \to +0$) applied to a system with free (or periodic) boundary. No external field is applied to Λ in the present proof.

If the configuration of $\{J_{ij}\}$ is fixed, the thermal average of the magnetization per site is calculated by

$$\langle M_\Lambda \rangle = 1 - 2\frac{\langle V_- \rangle}{N_\Lambda}, \tag{3.68}$$

where V_- is the total number of sites with down spins in a configuration of spins, and N_Λ is the size of Λ. We wish to obtain an upper bound to the configurational average of $\langle V_- \rangle$ as

$$[\langle V_- \rangle] \leq \frac{1-\alpha}{2} N_\Lambda, \tag{3.69}$$

where α is a constant independent of N_Λ. If (3.69) is proved, it readily follows that

$$[\langle M_\Lambda \rangle] \geq \alpha > 0, \tag{3.70}$$

see (3.68).

To prove (3.69), we first fix the configuration of spins and draw a border line between two neighboring spins if they are antiparallel to each other. It is evident that such borders form closed polygons (see Griffiths 1964 [13], 1972 [15]) because all spins on the boundary are up (no borders exist between two spins both on the boundary). For a given shape of border of length b, the total number of down spins it encloses is at most $(b/4)^2$ since the area which the border encloses is maximal when the border is a square. Thus, for a given configuration of spins, the total number of down spins is bounded as

$$V_- \leq \sum_{b \geq 4} \left(\frac{b}{4}\right)^2 \sum_{j=1}^{\nu(b)} X_b^j, \tag{3.71}$$

where $X_b^j = 1$ if a j-th border of length b is present in the spin configuration and $X_b^j = 0$ otherwise. The quantity $\nu(b)$ in (3.71) is the total number of possible borders of length b on Λ. The sum over b starts from four because a closed polygon has at least four edges. The average of (3.71) yields

$$[\langle V_- \rangle] \leq \sum_{b \geq 4} \left(\frac{b}{4}\right)^2 \sum_{j=1}^{\nu(b)} \left[\left\langle X_b^j \right\rangle\right]. \tag{3.72}$$

The next step of the proof is to estimate an upper bound to $\left[\left\langle X_b^j \right\rangle\right]$. In a fixed configuration of $\{J_{ij}\}$, the thermal average of X_b^j is calculated as

$$\begin{aligned} \left\langle X_b^j \right\rangle &= \frac{\sum_S X_b^j \exp(-\beta H)}{\sum_S \exp(-\beta H)} \\ &= \frac{\sum_S' \exp(-\beta H)}{\sum_S \exp(-\beta H)}, \end{aligned} \tag{3.73}$$

where the sum in the numerator is restricted to those states in which the j-th border is realized. An upper bound for (3.73) is obtained by restricting the sum over $\{S_i\}$ in the denominator to some special configurations of spins. Especially convenient configurations to be left in the denominator are the following ones. For each configuration C of spins in the numerator of (3.73) we associate another configuration C^* obtained therefrom by reversing all spins inside the j-th border of length b. Since a pair of spins neighboring across the border are anti-parallel to each other in C and parallel in C^*, we find

$$H(C^*) = H(C) - 2 \sum_{\text{border}} J_{ij}$$

where the sum runs over the border. By restricting the sum over $\{S_i\}$ in the denominator of (3.73) to $\{C^*\}$, we have an upper bound as

$$\left\langle X_b^j \right\rangle \leq \exp\left(-2\beta \sum_{\text{border}} J_{ij} \right). \tag{3.74}$$

The argument so far was a straightforward application of the Peierls argument to our random spin system. A new point of view is introduced by evaluating the configurational average of (3.74),

$$\left[\left\langle X_b^j \right\rangle \right] \leq \prod_{\text{border}} \left[\exp\left(-2\beta J_{ij} \right) \right]. \tag{3.75}$$

We first concentrate on the binary distribution (3.16). The above configurational average is then

$$[\exp(-2\beta J_{ij})] = p \exp(-2\beta J) + (1-p) \exp(2\beta J). \tag{3.76}$$

It is readily verified that (3.76) is equal to unity if $K_p = \beta J$. We thus set $c\beta J = K_p$ (c, a constant) and rewrite (3.76) as

$$[\exp(-2\beta J_{ij})] = \frac{\mathrm{e}^{-2\beta J} + \mathrm{e}^{-2(c-1)\beta J}}{1 + \mathrm{e}^{-2c\beta J}}. \tag{3.77}$$

Now that we have obtained an upper bound to $\left[\left\langle X_b^j \right\rangle \right]$, it is possible to estimate the right-hand side of (3.72). As is mentioned by Griffiths (1964 [13], 1972 [15]) an upper bound to $\nu(b)$ is given by $3^{b-1} N_\Lambda$. Therefore we have

$$[\langle V_- \rangle] \leq \sum_{b \geq 4} \left(\frac{b}{4} \right)^2 3^{b-1} N_\Lambda \frac{\left(\mathrm{e}^{-2\beta J} + \mathrm{e}^{-2(c-1)\beta J} \right)^b}{\left(1 + \mathrm{e}^{-2c\beta J} \right)^b}. \tag{3.78}$$

It is evident in (3.78) that $[\langle V_- \rangle]/N_\Lambda$ can be made arbitrarily small independently of the size of Λ if $c > 1$ and βJ is large enough. This completes the proof of (3.69) or (3.70) for the binary distribution.

Throughout the above proof we treated a system with a particular boundary condition (all spins on the boundary are up) in the absence of external field. The spontaneous magnetization is usually defined as

$$\lim_{h \to +0} \lim_{N \to \infty} M(h) \equiv M_S, \tag{3.79}$$

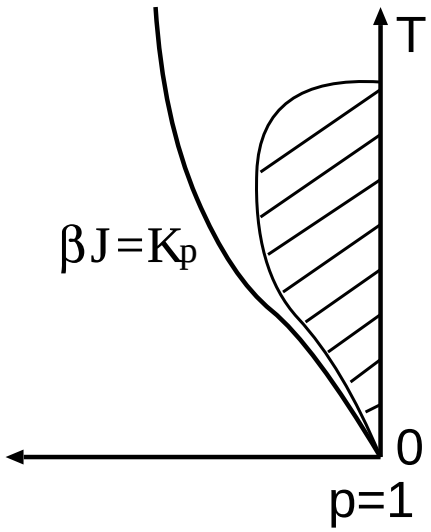

Figure 3.7:

where the system has a free boundary (or a periodic one). However, as was discussed by Griffiths (1972 [15]), the existence of the spontaneous magnetization (3.79) is equivalent to the existence of the magnetization with our special boundary condition. Intuitively this equivalence is trivial; both conditions (all spins up at the boundary, and the application of infinitesimal external field) pick out one of the two degenerate states brought about by ferromagnetic symmetry breakdown. A mathematical proof of the equivalence (Griffiths, 1972 [15]) for a non-random system makes use of the existence of thermodynamic limit of the free energy and the concavity of the free energy. Both these properties of the free energy are shared by random spin systems (Griffiths and Lebowitz 1968 [16], Griffiths 1972 [15]) and therefore the equivalence of the two definitions of magnetization is rigorously established even in random systems.

Now let us investigate physical implications of the result that the spontaneous magnetization exists if $c > 1$ (or $K_p > \beta J$) and βJ is large enough. On the phase diagram we have proved above the existence of the ferromagnetic phase in the hatched region of Figure 3.7. The curve $K_p = \beta J$ again plays a special role although we did not use gauge transformations in the present proof. We emphasize here that our failure to prove the existence of the spontaneous magnetization when $K_p = \beta J$ never means the absence of a ferromagnetic phase transition on the curve $K_p = \beta J$. "Cannot be proved" is not equivalent to "Can be disproved". In fact, if the spontaneous magnetization were absent on the curve $K_p = \beta J$, it would be concluded that the ferromagnetic phase is absent for any $p < 1$ and $0 < T < \infty$, according to the argument in section 3.6. This is clearly a contradiction. Therefore, on the square lattice, there should exist a ferromagnetic phase transition also along the curve $K_p = \beta J$. We notice that this existence on the special curve is a conclusion drawn by combining the result of the Peierls argument ($M_S > 0$ if $c > 1$ and $\beta J \gg 1$) and the discussion in section 3.6.

Analogous considerations lead to the same conclusion in higher dimensional lattices (see Griffiths, 1972 [15]). The result, $M_S > 0$ if $c > 1$ and $\beta J \gg 1$, remains unchanged.

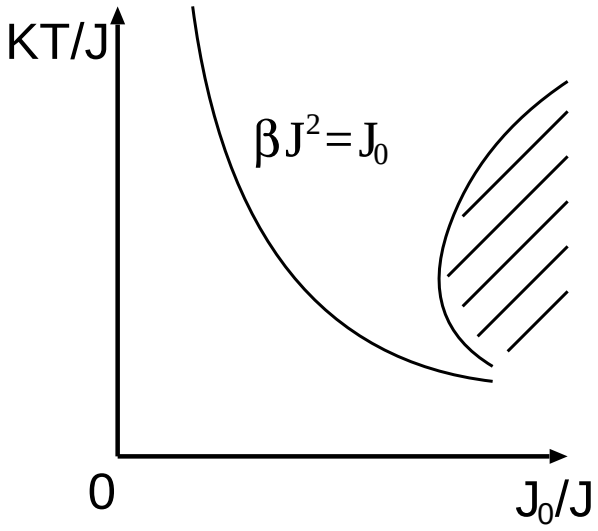

Figure 3.8:

Systems with other types of distribution functions also have corresponding ferromagnetic phases. A modification of the above proof is necessary only in the evaluation of the configurational average (3.75). For a general distribution,

$$[\exp(-2\beta J_{ij})] = \int_{-\infty}^{\infty} \mathrm{d}J_{ij} \exp(-2\beta J_{ij}) P(J_{ij}). \tag{3.80}$$

In the case of the Gaussian distribution (3.12), the above integration can be evaluated explicitly to yield

$$[\exp(-2\beta J_{ij})] = \exp(-2J_0\beta + 2J^2\beta^2). \tag{3.81}$$

If we set $c\beta J^2 = J_0$ (c, a constant), (3.81) is rewritten as

$$[\exp(-2\beta J_{ij})] = \exp\{(c^{-1} - 1)2\beta J_0\},$$

which becomes arbitrarily small if $c > 1$ and $\beta J_0 \gg 1$. The same considerations as those following (3.78) lead to the existence of the ferromagnetic phase at low temperatures in the region $c > 1$ (Figure 3.8). Analogous results are expected to hold for an arbitrary distribution function $P(J_{ij})$.

Chapter 4

Quantum Effects

So far we have concentrated on classical spin models. Real materials are of course quantum mechanical systems and we have to estimate quantum effects to construct a realistic theory. This chapter is devoted to a rigorous analysis of a special kind of random quantum spin systems.

4.1 Impossibility of gauge transformation

In the previous chapter the gauge transformation (3.4) was shown to play a crucial role in obtaining rigorous results on classical systems. The gauge transformation was, in a word, a change of the sign of a spin variable. This change of the sign of a spin variable is impossible in quantum mechanical systems; spin operators should satisfy the commutation relation

$$[S_j^x, S_j^y] = iS_j^z \tag{4.1}$$

which is not left invariant by the simultaneous change of signs of all components $(S_j^x, S_j^y, S_j^z) \to (-S_j^x, -S_j^y, -S_j^z)$. In fact (4.1) is then transformed into

$$[S_j^x, S_j^y] = -iS_j^z \tag{4.2}$$

which is clearly different from (4.1). Equation (4.2) can be derived from (4.1) also by changing i to $-i$, which is equivalently said to be the time reversal operation (Messiah, 1964 [30]). Time reversal is an allowed operation of a system, but our point is that the operation is impossible to be performed at each site independently of other sites. Time reversal at one point of the space is a senseless idea, while our gauge transformation must be a local one to characterize random spin systems as has been emphasized so far. Therefore we conclude that a gauge transformation is generally impossible to be defined in quantum mechanical systems[1].

This impossibility of local gauge transformation is well illustrated by the ground-state of the antiferromagnetic Heisenberg model on a bipartite lattice (a lattice with two

[1]The XY-model is an exception. In the XY-model the Hamiltonian is written in terms of the x and y components of spin operators. Simultaneous change of the x and y components (z component unchanged) leaves the commutation relation invariant, and hence a gauge transformation is consistently defined.

sublattices). If the spin variables are classical, the Néel state is the ground-state. Neighboring spins of a spin are rigorously antiparallel to the spin which they are surrounding. However if the spins are quantum operators, the Néel state is not necessarily a good approximation to the ground-state wave function (Suzuki and Miyashita, 1978 [40]). In other words, we cannot derive the antiferromagnetic ground-state from the ferromagnetic one just by reversing spins on one of the sub lattices. This can be alternatively said to be the impossibility of gauge transformation.

The impossibility of local gauge transformation throws doubt on the validity of the concept of frustration. In the classical case thermodynamic properties of the Mattis model, for instance, were proved to be exactly equivalent to those of a pure ferromagnetic system and thus be irrelevant to spin glass phenomena — all because the Mattis model is free of frustrations. In a quantum mechanical version of the Mattis model, however this may not be the case. Consider a system with the Hamiltonian

$$H = -J \sum \tau_i \tau_j \mathbf{S}_i \cdot \mathbf{S}_j \qquad (J > 0), \tag{4.3}$$

where $\tau_i (= \pm 1)$ is an arbitrarily fixed random parameter and $\mathbf{S}_i$ is a quantum spin operator. If $\mathbf{S}_i$ is a classical vector, a simple re-definition of the variable $\tau_i \mathbf{S}_i \to \mathbf{S}_i$ changes (4.3) to a pure system. As was discussed above, this transformation is not allowed in a quantum mechanical system. It follows that the system (4.3) may behave quite differently from a pure one. If this is the case, the concept of frustration does not make sense and the classical classification of randomness, the frustrated and the unfrustrated, is invalidated.

Luck favored us, and we have found that the situation is not so hopeless; we prove in the rest of this chapter that the classical picture on the ground-state of a classically frustration-free system is not totally invalid, at least as concerns the total spin quantum number. In fact, since a classical model is obtained as the $S \to \infty$ limit of a quantum system, no serious contradiction between the classical and quantum mechanical pictures is expected to exist if S is large enough.

4.2 Ground-state symmetry of the isotropic Mattis-Heisenberg model

With the remarks in the previous section in mind, we investigate the ground-state of the following system:

$$H = -J \sum \tau_i \tau_j (S_i^x S_j^x + S_i^y S_j^y + \Delta S_i^z S_j^z) \quad (J > 0,\ -\infty < \Delta < \infty,\ \tau_i = \pm 1). \tag{4.4}$$

The lattice structure is not specified. We prove that the ground-state of this system is unique except for trivial degeneracies listed below. Moreover we rigorously calculate the total spin quantum number of the ground-state wave function. The result is classified by the value of Δ:

(i) $-1 < \Delta < 1$, $\quad M_z = \sum_i S_i^z = 0$ (if N_S is an integer)
$= \pm 1/2$ (if N_S is a half-integer),
where S is the magnitude of a spin at each site.

(ii) $\Delta > 1$, $\quad M_z = \pm\left(\sum_i \tau_i\right) S$

(iii) $\Delta = 1$, $\quad S_{\rm total} = |\sum_i \tau_i| S,\ M_z = -S_{\rm total},\ -S_{\rm total}+1,\ \cdots S_{\rm total}$.

(iv) $\Delta = -1$, $\quad S_{\rm total} = |\sum_i \tau_i'| S,\ M_z = -S_{\rm total},\ -S_{\rm total}+1,\ \cdots S_{\rm total}$.
This formula is valid only when the lattice is bipartite. The set $\{\tau_i'\}$ here is obtained from the original $\{\tau_i\}$ by changing the signs of all τ_i on one of the sublattices.

(v) $\Delta < -1$, $\quad M_z = \pm\left(\sum_i \tau_i'\right) S$ if the lattice is bipartite. The set $\{\tau_i'\}$ here is the same one as in (iv).

Our proof of these results is a generalization of the argument by Mattis (1979 [29]); he proved that the spin quantum number in the ground-state of the pure ferromagnetic XY-model ($\Delta = 0$ and all $\tau_i = 0$ in (4.4)) is given by the above statement (i). The most important part of his proof is the construction of a "reference Hamiltonian" with long-ranged interactions. This strategy is inherited by our proof; we demonstrate that the symmetry of the ground-state of the Hamiltonian (4.4) is essentially equivalent to that of an infinite-ranged system which is easily solved.

To make the representation transparent, we first perform a canonical transformation

$$S_i^x \to \tau_i S_i^x, \quad S_i^y \to -S_i^z \quad S_i^z \to \tau_i S_i^y. \tag{4.5}$$

This transformation is easily verified to preserve the commutation relation (4.1) (and two similar equations obtained from (4.1) by cyclic changes of x, y and z). By the transformation (4.5) the Hamiltonian (4.4) is changed into

$$H = -\frac{J}{4}\sum \left\{(1-\Delta)(S_i^+ S_j^+ + S_i^- S_j^-) + (1+\Delta)(S_i^+ S_j^- + S_i^- S_j^+ + 4\tau_i\tau_j S_i^z S_j^z)\right\}, \tag{4.6}$$

where $S_j^{\pm} = S_j^x \pm {\rm i} S_j^y$. We notice that the signs of the first two terms on the right-hand side of (4.6) change at $\Delta = 1$ and $\Delta = -1$. It will be seen that these changes of signs are closely related to the classification from (i) to (v) listed above. In the original expression of the Hamiltonian (4.4), the z-component M_z of the total spin was a good quantum number. In the new representation (4.6), $\sum_i \tau_i S_i^y$ commutes with the Hamiltonian as is evident from the transformation (4.5), $\sum_i S_i^z \to \sum_i \tau_i S_i^y$. Therefore we calculate the eigenvalue of the operator $\sum_i \tau_i S_i^y$ in the ground-state of (4.6). If $\Delta = 1$, the magnitude of the total spin itself is also a good quantum number as one is convinced by the representation (4.4).

In (4.6), the operation of any term of the Hamiltonian changes the eigenvalue of $\sum_i S_i^z$ by an even number, two or zero. Accordingly the space of states is decoupled to two subspaces, the even and the odd. These subspaces are spanned by

$$\phi_{\rm even} = C\prod_i (S_i^+)^{p_i}|0\rangle, \quad \sum_i p_i = \text{even} \tag{4.7}$$

and

$$\phi_{\rm odd} = C\prod_i (S_i^+)^{p_i}|0\rangle, \quad \sum_i p_i = \text{odd} \tag{4.8}$$

respectively. In these equations, C denotes the normalization constant, p_i is a non-negative integer, and $|0\rangle$ represents the state in which $S_i^z = -S$ for all i. We proceed to seek the ground-state of each subspace and then determine the true ground-state of the whole system.

Let ψ be the ground-state wave function in one of the subspaces. ψ is expanded by the bases of the form (4.7) or (4.8):

$$\psi = \sum_\nu a_\nu \phi_\nu \tag{4.9}$$

where ν runs over all states in one of the subspaces and the coefficients $\{a_\nu\}$ are supposed to satisfy the normalization condition

$$\sum_\nu |a_\nu|^2 = 1. \tag{4.10}$$

To find out a necessary condition for ψ to be the ground-state, we calculate the expectation value of the Hamiltonian (4.6) by ψ and then minimize the expectation value. By using the expression (4.9) of ψ, we find

$$\begin{aligned} E &\equiv \langle\psi|H|\psi\rangle \\ &= -\frac{J}{4}(1-\Delta)\sum\nolimits_\nu \sum\nolimits_\mu{}' |a_\nu||a_\mu| \cos(\theta_\nu - \theta_\mu) \\ &\quad -\frac{J}{4}(1+\Delta)\sum\nolimits_\nu \sum\nolimits_\lambda{}' |a_\nu||a_\lambda| \cos(\theta_\nu - \theta_\lambda) \\ &\quad +(\text{diagonal terms}), \end{aligned} \tag{4.11}$$

where θ_ν is the phase of a_ν and the "diagonal terms" mean the expectation value of the third term on the right-hand side of (4.6) (expressed in terms of S_i^z). In (4.11) the sums over μ and λ are restricted to the following states: μ is a state obtained from ν by the operation of $S_i^+ S_j^+ + S_i^- S_j^-$, and λ is obtained from ν by $S_i^+ S_j^- + S_i^- S_j^+$. Since the Hamiltonian operator (4.6) is Hermitian, the expectation value of it, the internal energy, is real. For this reason only the real part is left in (4.11).

It is noticed that the third term $\tau_i \tau_j S_i^z S_j^z$ on the right-hand side of (4.6) is diagonal in the representation (4.7) or (4.8). Thus the "diagonal terms" in (4.11) are functions only of the absolute values $|a_\nu|$ of the coefficients. On the other hand a necessary condition for ψ to be the ground-state is obtained if we minimize the energy expectation value (4.11) by varying $\{\theta_\nu\}$ arbitrarily with the absolute value fixed. A condition on the phases $\{\theta_\nu\}$ thus obtained will be found to be sufficient to determine the spin quantum number in the ground-state. Consequently the "diagonal terms" in (4.11) play no roles in the following arguments.

Now let us investigate each value of Δ.

(i) $-1 < \Delta < 1$

If Δ is between -1 and $+1$, both coefficients $J(1-\Delta)/4$ and $J(1+\Delta)/4$ in (4.11) are positive. In this case, to minimize the energy expectation value (4.11), we have to choose all θ_ν equal to a common value because the cosine factors, $\cos(\theta_\nu - \theta_\mu)$ and $\cos(\theta_\nu - \theta_\lambda)$, are

then equal to unity. One may suspect that the value of phases is a constant only within a set of such states as are obtained from ν by $S_i^+S_j^+ + S_i^-S_j^-$ or $S_i^+S_j^- + S_i^-S_j^+$. This suspicion is based on the observation that the phases appear in (4.11) only as combinations like $\cos(\theta_\nu - \theta_\mu)$ and $\cos(\theta_\nu - \theta_\lambda)$ with μ and λ defined as before. To extremize these cosines, it apparently seems enough to set $\theta_\nu = \theta_\mu$ and $\theta_\nu = \theta_\lambda$ only for μ and λ obtained from ν by $S_i^+S_j^+ + S_i^-S_j^-$ and $S_i^+S_j^- + S_i^-S_j^+$. However we have to set *all* θ_ν's equal to a common constant to minimize (4.11), because, in a word ,the off-diagonal terms in (4.6) (terms including S_i^+ and S_i^-) connect *all* states in one of the subspaces in the following sense. If a μ is derived from ν, then $\theta_\nu = \theta_\mu$. And this μ appears as another ν in the summation over ν in (4.11), making another group of phases $\{\theta_\mu\}$ equal to θ_μ. By repeating this procedure we find that all θ_ν are equal to each other since this procedure connects all states in a subspace. A common phase factor is irrelevant to physics, and we may simply choose it to be zero. In this way we have found that all coefficients a_ν in (4.9) are real and non-negative. Furthermore we can show that none of the coefficients are vanishing, that is, they are all positive definite.

To prove $a_\nu > 0$, we write the eigenvalue equation $H\psi = E\psi$ projected to a state ν (Lieb and Mattis, 1962 [24]):

$$-\frac{J}{4}\sum_\mu{}'(1-\Delta)|a_\mu| - \frac{J}{4}\sum_\lambda{}'(1+\Delta)|a_\lambda| + f_\nu|a_\nu| = Ea_\nu, \tag{4.12}$$

where we have made use of the fact $\theta_\nu = 0$ (for all ν). The summations over μ and ν are restricted to the states explained in the previous paragraph. In (4.12) the term $f_\nu|a_\nu|$ corresponds to the "diagonal terms" in (4.11). If $|a_\nu| = 0$, it is seen in (4.12) that $|a_\mu| = |a_\lambda| = 0$ because $J(1-\Delta) > 0$ and $J(1+\Delta) > 0$. Repeated application of this argument reveals $a_\nu = 0$ for all states in a subspace, since the sequence of states thus generated includes all states as was discussed before. As the wave function is normalized (4.10), it is impossible for all a_ν to vanish. Hence none of the coefficients can be zero. We have thus shown that all $a_\nu > 0$ in (4.9). This is a necessary condition of the ground-state wave function.

It readily follows that the ground-state is unique in each subspace. Suppose that ψ and ψ' are two ground-states in a subspace. They can always be chosen to be orthogonal to each other since they are degenerate (see, *e. g.*, Courant and Hilbert, 1962 [7]). Both ψ and ψ' should satisfy $a_\nu > 0$ in the expansion (4.9) for they are ground-state wave functions. This is a contradiction because $\langle\psi|\psi'\rangle \neq 0$ if both ψ and ψ' have positive definite coefficients. Therefore we conclude that the ground-state is unique in each subspace (even and odd).

Next let us consider an infinite ranged system (the reference system)

$$H_{\rm ref} = -\frac{1}{N}\left\{\left(\sum_i S_i^x\right)^2 + \left(\sum_i S_i^z\right)^2\right\}, \tag{4.13}$$

which will be shown to have the same spin quantum number in the ground-state as our system (4.6). The eigenvalue of the Hamiltonian (4.13) is easily obtained as

$$H_{\rm ref} = -\frac{1}{N}\left\{I(I+1) - m^2\right\}, \tag{4.14}$$

where I is the total spin quantum number

$$I(I+1) = \left(\sum_i \vec{S}_i \right)^2 ,$$

and m denotes the y-component of the total spin

$$m = \sum_i S_i^y .$$

The total spin quantum number can take on the following values

$$\begin{aligned} I &= NS,\ NS-1,\ NS-2,\ \cdots,\ 1,\ 0 \quad \text{if } NS \text{ is an interger. (a)} \\ &= NS,\ NS-1,\ NS-2,\ \cdots,\ 1,\ 0 \quad \text{if } NS \text{ is a half-interger. (b)} \end{aligned} \tag{4.15}$$

Correspondingly the y-component m is possible to assume

$$m = I,\ I-1,\ \cdots\ -I .$$

It is evident in (4.14) that the ground-state quantum numbers are $I = NS$, $m = 0$ if NS is an integer, or $I = NS$, $m = \pm 1/2$ if NS is a half-integer.

In order to make it possible to compare the original system (4.6) with the reference system (4.13), we should perform a canonical transformation $S_i^y \to \tau_i S_i^y$ $S_i^z \to \tau_i S_i^z$ in (4.13), because the original and reference systems then commute with a common operator $\sum_i \tau_i S_i^y$.[2] Since a canonical transformation does not change the eigenvalues of operators, we conclude the transformed reference system

$$H'_{\text{ref}} = -\frac{1}{N} \left\{ \left(\sum_i S_i^x \right)^2 + \left(\sum_i \tau_i S_i^z \right)^2 \right\} \tag{4.16}$$

has the ground-state spin quantum number

$$m = \sum_i \tau_i S_i^y = 0 \ \text{ or } \ \pm 1/2.$$

To show the essential equivalence of the ground-state properties of the two systems (4.6) and (4.16), we derive a necessary condition for a wave function of the form (4.9) to be the ground-state of the transformed reference system (4.16). Equation (4.16) can be written as

$$H'_{\text{ref}} = -\frac{1}{4N} \sum_{i,j} \left(S_i^+ S_j^+ + S_i^- S_j^- + S_i^+ S_j^- + S_i^- S_j^+ \right) - \frac{1}{N} \sum_{i,j} \tau_i \tau_j S_i^z S_j^z . \tag{4.17}$$

By comparison with (4.6) ($-1 < \Delta < 1$), the ground-state wave function of (4.17) is concluded to satisfy $a_\nu > 0$ for all coefficients in the expansion (4.9) (quite the same

[2]The reference system (4.13) commutes with $\sum_i S_i^y$ which is transformed into $\sum_i \tau_i S_i^y$ by the above canonical transformation.

argument as that in the foregoing several pages applies also to the long-ranged model (4.17) to yield $a_\nu > 0$). It follows again that the ground-state of each subspace is unique.[3]

First consider the case NS = integer. The ground-state spin quantum number of (4.17) is then $m = 0$, clearly non-degenerate. Hence the ground state of the whole system ψ_g is in one of the subspaces. The ground-state ψ'_g of (4.6) in the same subspace as that of ψ_g is not orthogonal to ψ_g because both ψ_g and ψ'_g have positive coefficients in the expansion in the same subspace. Therefore the ground-states ψ_g and ψ'_g share the same quantum number $m = 0$. (If they had different quantum numbers, they should be orthogonal to each other.)

If NS is a half-integer, the ground-state quantum number of the long-ranged system (4.17) is $m = \pm 1/2$, doubly degenerate. As was remarked before, however, the ground-state of each subspace has no degeneracy. The only possible way out is that the ground-states of the two subspaces are degenerate to each other (the ground-state energy of the even space = that of the odd space). We also notice that the operator $\sum_i \tau_i S_i^y$, whose eigenvalue is denoted by m, connects the two subspaces (recall $S_j^y = (S_j^+ - S_j^-)/2\mathrm{i}$). Therefore the ground-states of the whole system (of the infinite-ranged model) should be the following linear combinations:

$$\psi_\pm = \psi_{\text{even}} \pm \psi_{\text{odd}}, \tag{4.18}$$

where ψ_{even} and ψ_{odd} are the ground-states of the subspaces. ψ_{even} and ψ_{odd} have the same energy eigenvalue. Let us construct counterparts of (4.18) in the (arbitrarily ranged) system (4.6):

$$\psi'_\pm = \psi'_{\text{even}} \pm \psi'_{\text{odd}}, \tag{4.19}$$

where ψ'_{even} and ψ'_{odd} are the ground-states of the two subspaces of (4.6). Since all of the four subspace ground-states in (4.18) and (4.19) have positive definite coefficients, (4.18) cannot be orthogonal to (4.19), that is $\langle\psi_+|\psi'_+\rangle \neq 0$ and $\langle\psi_-|\psi'_-\rangle \neq 0$. This non-orthogonality readily implies the same quantum number $m = \pm 1/2$ for ψ' as $m = \pm 1/2$ for $\psi_\pm$. This completes the proof of the statement (i).

(ii) $\Delta > 1$

When $\Delta > 1$, the coefficient $J(1-\Delta)/4$ of the first term on the right-hand side of (4.11) is negative. The rule for the phase θ_ν to minimize (4.11) is modified as follows. To make the second term on the right-hand side of (4.11) minimal, we should choose all θ_ν = constant within a subspace of a definite $\sum_i S_i^z$. (Recall that λ in (4.11) was obtained by the operation of $S_i^+ S_j^- + S_i^- S_j^+$ which does not change $\sum_i S_i^z$). On the other hand, to minimize the first term on the right-hand side of (4.11), it is necessary to set $\theta_\mu = \theta_\nu + \pi$, for $J(1-\Delta) < 0$. We notice that μ is obtained from ν by $S_i^+ S_j^+ + S_i^- S_j^-$ which changes $\sum_i S_i^z$ by ± 2. Therefore the phase rule is: $\theta_\nu = 0$ for those ν whose eigenvalues of $\sum_i S_i^z$ are different (from the corresponding eigenvalue of an arbitrarily chosen state ν_0) by multiples of four, and $\theta_\nu = \pi$ for those ν which differ in $\sum_i S_i^z$ from ν_0 by odd multiples of two. For instance, we can chose $\theta_\nu = 0$ for ν satisfying $\sum_i S_i^z = -NS, -NS+4, \cdots$ and $\theta_\nu = \pi$ for such states as $\sum_i S_i^z = -NS+2, NS+6 \cdots$. Since we are considering

[3]The space of states of (4.17) also decouples into the even and odd subspaces.

in one of the subspaces, no sates with $\sum_i S_i^z = -NS+1, -NS+3, \cdots$ appear in the above list.

It is also possible to prove that no coefficients vanish. The argument proceeds quite similarly as before. The eigenvalue equation corresponding to (4.12) is

$$\frac{J}{4}\sum_{\mu}{}'(1-\Delta)|a_\mu| - \frac{J}{4}\sum_{\lambda}{}'(1+\Delta)|a_\lambda| + f_\nu|a_\nu| = E|a_\nu|, \tag{4.20}$$

where the phase rule, $\theta_\mu = \pi$ if $\theta_\nu = 0$, has been taken into account. Equation (4.20) proves that, if $|a_\nu| = 0$, all of $|a_\mu|$'s and $|a_\lambda|$'s are vanishing because $J(1-\Delta) < 0$ and $J(1+\Delta) > 0$. Repeated application of this procedure yields $a_\nu = 0$ for all states, an evident contradiction. Before we write down a reference system, we mention that the ground-state in each subspace is unique, a result obtained by the orthogonality consideration given in the proof of (i)

An appropriate reference system is

$$H_{\rm ref} = -\frac{1}{N}\left(\sum_i S_i^y\right)^2, \tag{4.21}$$

which evidently commutes with $\sum_i \tau_i S_i^y$. Thus we look for the eigenvalue m of $\sum_i \tau_i S_i^y$ in the ground-state of (4.21). The ground state of the reference system (4.21) is clearly doubly degenerate, $S_i^y = +S$ for all i and $S_i^y = -S$ for all i. The quantum number is thus $m = \pm(\sum_i \tau_i)S$. To compare with the system (4.6), we expand (4.21) as

$$H_{\rm ref} = \frac{1}{4N}\sum_{i,j}\left(S_i^+S_j^+ + S_i^-S_j^- - S_i^-S_j^+ - S_i^+S_j^-\right). \tag{4.22}$$

This representation of the reference system has quite a similar structure to (4.6) with $\Delta > 1$ in that the signs of the coefficients of spin operators are the same. Since the diagonal terms $\tau_i\tau_j S_i^z S_j^z$ in (4.6) have nothing to do with the phase rule, we conclude that the same phase rule as the previous one ($\theta_\nu = 0$, $\theta_\mu = \pi$) applies to the ground-state of the infinite-ranged system (4.22). Uniqueness of the ground-state in each space, even or odd, follows also for the reference system.

Now that we have found out the ground-state quantum number $m = \pm(\sum_i \tau_i)S$ and the uniqueness of the ground-state in a sub-space, we can proceed in the same way as in (i) to prove the essential equivalence of the infinite-ranged system and the general (arbitrarily ranged) system. Since the ground-state of the whole system of the reference model is doubly degenerate (except when $\sum_i \tau_i = 0$), we construct the wave function $\psi_\pm$ as in (4.18) by taking account of the degeneracy of the even and odd space ground-states. The corresponding linear combinations of the subspace ground-states of the general system are also constructed like (4.19) and are denoted as $\psi'_\pm$. Since all subspace ground-states satisfy the same phase rule, ψ_+ cannot be orthogonal to ψ'_+ and ψ_- cannot be to ψ'_-. Therefore $\psi_\pm$ and $\psi'_\pm$ share the same quantum number. We have thus proved the statement (ii)

(iii) $\Delta = 1$
If $\Delta = 1$, the first term on the right-hand side of (4.6) vanishes. This means a further

decoupling of the space of states. In fact, in the original representation (4.4), it is evident that the system with $\Delta = 1$ has another spin quantum number, magnitude of the total spin. This case needs a special consideration. A compact and comprehensive argument is already available in the paper by Lieb and Mattis (1962 [24]). We Just refer to their result here. Our system (4.4) corresponds to the special case $g^2 = 0$ in their notation. Sites with $\tau_i = +1$ in our Hamiltonian constitute the A sublattice in their paper and those with $\tau_i = -1$ form the B sublattice. Thus $|S_A - S_B|$ in their Eqn.(3c) is equal to $|\sum_i \tau_i| S$ which is the ground-state total spin quantum number. States with $M_z = -S_{\rm total}, \cdots, +S_{\rm total}$ are degenerate because the total spin is a good quantum number.

(iv), (v) $\Delta \leq -1$
If the lattice is bipartite, it is possible to rewrite the Hamiltonian with $\Delta \leq 1$ into that with $\Delta \geq 1$. First, let us change the signs of all τ_i's on one of the sublattices. The Hamiltonian (4.4) is then transformed into

$$H = J \sum \tau_i \tau_j (S_i^x S_j^x + S_i^y S_j^y - |\Delta| S_i^z S_j^z). \tag{4.23}$$

We perform a canonical transformation, $S_i^x \to -S_i^x$ and $S_i^y \to -S_i^y$ on one of the sublattices. The result is

$$H = -J \sum \tau_i \tau_j (S_i^x S_j^x + S_i^y S_j^y + |\Delta| S_i^z S_j^z). \tag{4.24}$$

We have thus proved the equivalence to $|\Delta| \geq 1$, and the statements (iv) and (v) readily follow from (ii) and (iii). We can prove nothing when the lattice is not bipartite and $\Delta \leq -1$. Classically this is a case with frustration. No rigorous results on classically frustrated quantum systems have been obtained.

4.3 Discussion

One of the remarkable features of the statements proved in the foregoing section is the discontinuous changes of quantum numbers at $\Delta = 1$ and $\Delta = -1$. These changes are in accordance with the concept of universality class. Roughly speaking, we may say that systems with $-1 < \Delta < 1$ behave like the XY-model $\Delta = 0$, those with $\Delta > 1$ are similar to the Ising model $\Delta \to \infty$, and $\Delta = 1$ constitutes a special universality class with higher symmetry (see Jasnow and Wortis 1968 [20], Suzuki and Takano 1979 [41] and Takano and Suzuki 1981 [42] for detailed descriptions of the relevant universality classes). Our conclusion here on this problem is that the concept of universality class is valid even in the presence of site randomness, as far as the ground-state spin quantum number is concerned. Therefore the site randomness in the anisotropic Mattis-Heisenberg model (4.4) does not cause any essential changes from a pure ferromagnetic system.

Moreover the values of the spin quantum numbers imply irrelevance of quantum effects in the following sense. If the spin variables are treated classically in (4.4) with $\Delta > 1$, the spin configuration in the ground-state is evidently $S_i^z = \tau_i S$ for all i or $S_i^z = -\tau_i S$ at all i. In these configurations, $\sum_i S_i^z$ is equal to $\pm \sum_i \tau_i S_i$. This result coincides with the statement (ii). Therefore we may say that the ground-state is not very far from the

classical state — each spin is pointing either up or down depending only on the value of τ_i of the site. If $|\Delta| < 1$, the statement (i) reveals that the site randomness does not contribute at all to the determination of the quantum number. The statement (i), $M_z = 0$ or $\pm 1/2$, simply supports the classical picture that the total spin lies on the xy-plane. We conclude from these considerations that the classical picture should not be far from the real state.

One may further suppose that the site randomness in the anisotropic Mattis-Heisenberg model has no serious effects on thermodynamics of the system. The thermal properties of the Mattis-Heisenberg model may actually be the same to those of a pure system as was the case classically. However we have to be careful not to mention anything uncertain; what we have done is only to calculate the ground-state total spin and prove the uniqueness (except trivial degeneracies) of the ground-state. Nothing rigorous has been proved pertaining to excited states and finite temperature properties. Still, the result stated from (i) to (v) above is encouraging since it shows that quantum effects do not change the situation into something contradictory to the concept of frustration.

4.4 Frustrated quantum systems

We have thus far investigated the anisotropic Mattis-Heisenberg model which is classically free of frustrations. It is very difficult to derive rigorous results on classically frustrated systems. No canonical transformation can transform such systems to apparently non random ones in the same way as we changed the Mattis-Heisenberg model (4.4) into (4.6). In this sense classically frustrated systems have essential randomness even when treated quantum mechanically. Hence it is reasonable to call them frustrated quantum systems. There are a few efforts on frustrated quantum systems to establish a physical picture of them, for instance by investigating finite systems (Marland and Betts, 1979 [26]) and by heuristic arguments (Fazekas 1980 [10], Fazekas and Anderson 1974 [11]). However it is far beyond our present ability to construct a general theory of these systems.

Chapter 5

Summary and concluding remarks

We have investigated the significance of gauge transformation. The concept of frustration was defined in classical spin systems to extract relevant randomness to spin glass states. Gauge transformation was shown to be a dual notation of frustration. Invariance under gauge transformation characterized a class of physical quantities and this invariance property was demonstrated to be shared by the lattice gauge theory of strong interactions. Gauge transformation is extremely useful in obtaining a variety of rigorous results on thermodynamic quantities. For instance, in a subspace of the phase diagram, the exact value of the internal energy, an upper bound to the specific heat and several correlation inequalities were derived. Quantum effects make classically trivial systems non-trivial. The concept of frustration is somewhat vague quantum mechanically. It was shown, however, that the classical picture on the ground-state of a classically frustration-free system is valid at least concerning the total spin quantum number.

By our method in this thesis it is impossible to prove or disprove existence of the spin glass phase as a thermodynamically stable state. The spin glass phase is an ordered state along the time axis, and its existence would be closely related to the ergodicity. The present analysis of symmetries is still poor to fully follow up the ergodicity of random systems.

In chapter 3 the subspace $A_{ij} = \beta J_{ij}$ was shown to play a crucial role in obtaining rigorous results. In this subspace many physical quantities behaved very trivially as if the systems were non-interacting ones. A deeper insight may reveal an important physical significance of the subspace.

Frustrated quantum systems at present refuse any rigorous analysis (except the infinite ranged case solved by Bray and Moore, 1980 [5]). To prove some definite statements on these systems is one of the most challenging problems in future.

Appendix A

Proof of $q = m$ on the line $\beta\tilde{J}^2 = \tilde{J}_0$ in long-ranged Gaussian model

According to Sherrington and Kirkpatrick (1975 [37]), the order parameters q and m are determined by the following equations:

$$q = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-z^2/2} \tanh^2 \left\{ \sqrt{q}\beta\tilde{J}z + \beta m\tilde{J}_0 \right\} \tag{A.1}$$

$$m = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-z^2/2} \tanh \left\{ \sqrt{q}\beta\tilde{J}z + \beta m\tilde{J}_0 \right\}. \tag{A.2}$$

We verify that assumption $q = m$ is consistent with this set of equations if $\beta\tilde{J}^2 = \tilde{J}_0$. If we write $\sqrt{q}\beta\tilde{J} = a$, then $\beta m\tilde{J}_0 = \beta^2\tilde{J}^2 q = a^2$ under the present assumption. Equations (A.1) and (A.2) are then

$$q = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-z^2/2} \tanh^2(az + a^2) \tag{A.3}$$

$$m = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-z^2/2} \tanh(az + a^2). \tag{A.4}$$

Therefore we have only to prove that the right-hand sides of (A.3) and (A.4) are equal to each other for any a. Let us rewrite (A.3) into a convenient form to compare with (A.4):

$$\begin{aligned}
(2\pi)^{1/2} q &= \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-(z-a)^2/2} \tanh^2 az \\
&= \mathrm{e}^{-a^2/2} \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-z^2/2+az} \tanh^2 az \qquad \text{(A.5)} \\
&= \mathrm{e}^{-a^2/2} \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-z^2/2-az} \tanh^2 az \qquad \text{(A.6)}
\end{aligned}$$

We sum (A.5) and (A.6) then divide by 2:

$$\begin{aligned}
(2\pi)^{1/2} q &= \mathrm{e}^{-a^2/2} \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{z^2/2} \cosh az\, \tanh^2 az \\
&= \mathrm{e}^{-a^2/2} \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{z^2/2} \sinh^2 az\, \mathrm{sech}\, az. \qquad \text{(A.7)}
\end{aligned}$$

An analogous manipulation can be applied to (A.4):

$$\begin{aligned}
(2\pi)^{1/2} m &= \int_{-\infty}^{\infty} \mathrm{d}z \, \mathrm{e}^{-(z-a)^2/2} \tanh az \\
&= \mathrm{e}^{-a^2/2} \int_{-\infty}^{\infty} \mathrm{d}z \mathrm{e}^{-z^2/2+az} \tanh az \\
&= \mathrm{e}^{-a^2/2} \int_{-\infty}^{\infty} \mathrm{d}z \mathrm{e}^{-z^2/2-az} \tanh az \\
&= \mathrm{e}^{-a^2/2} \int_{-\infty}^{\infty} \mathrm{d}z \mathrm{e}^{-z^2/2} \sinh az \, \tanh az \\
&= \mathrm{e}^{-a^2/2} \int_{-\infty}^{\infty} \mathrm{d}z \mathrm{e}^{-z^2/2+az} \sinh^2 az \, \mathrm{sech}\, az. \qquad \text{(A.8)}
\end{aligned}$$

Equations (A.8) and (A.8) prove $q = m$.

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